Lecture 18: Glass

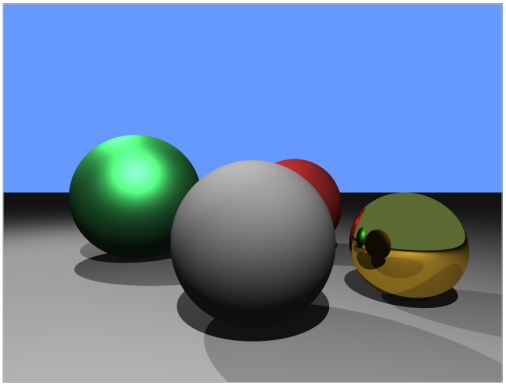
Snell's Law, the Bent Ray, and Rays That Split

CS 453 — Computer Graphics

Fakhir Shaheen

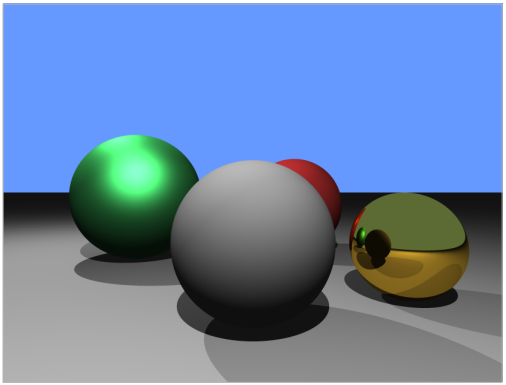
Forman Christian University

Today: the front ball becomes glass



a chalk ball in front

Today: the front ball becomes glass



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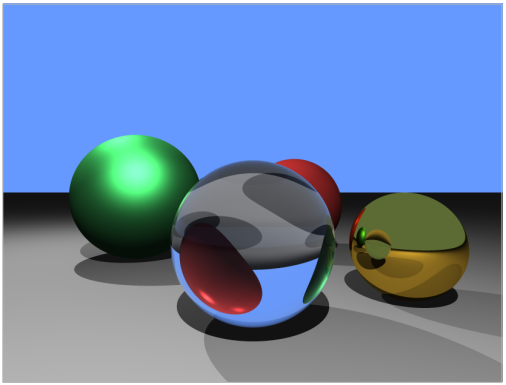
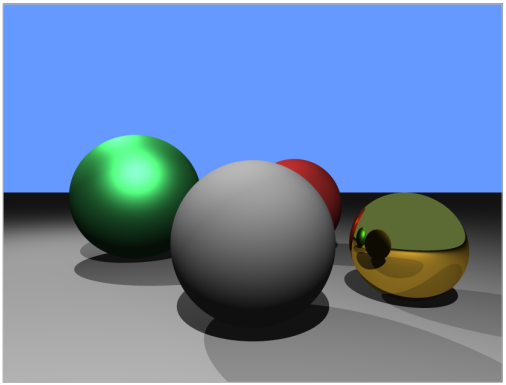


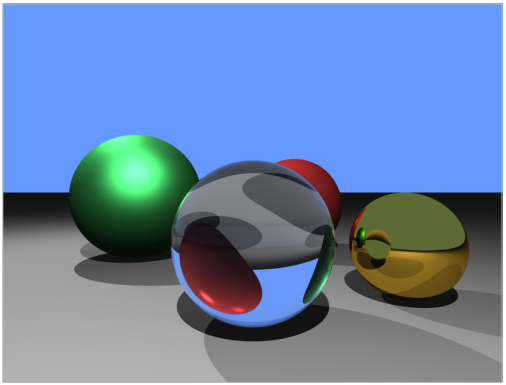
the same ball, made of glass

- We see through the ball

Today: the front ball becomes glass



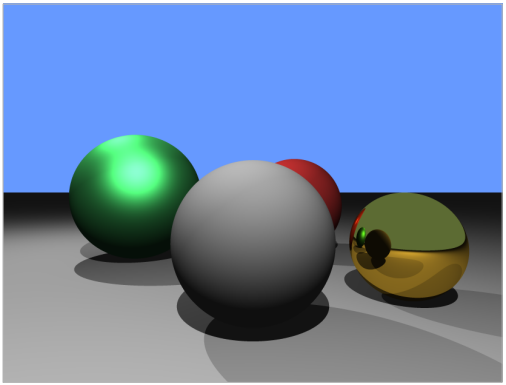
a chalk ball in front



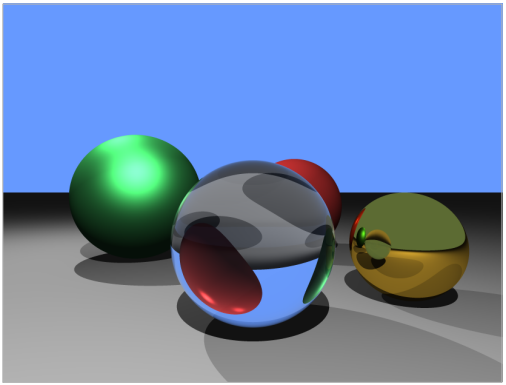
the same ball, made of glass

- We see through the ball
- Inside it: upside down

Today: the front ball becomes glass



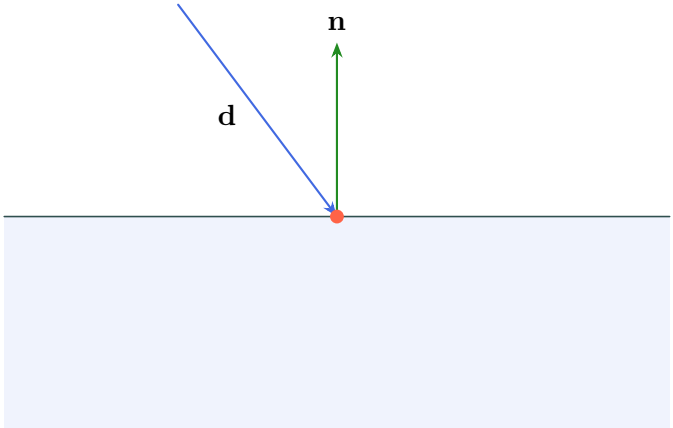
a chalk ball in front



the same ball, made of glass

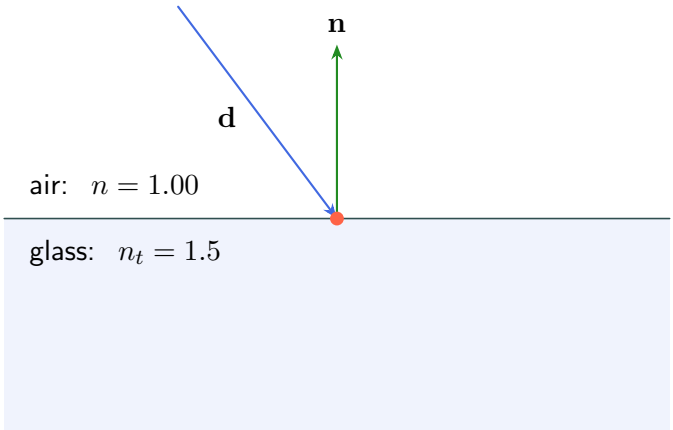
- We see through the ball
- Inside it: upside down
- Its rim: a mirror

Our conventions in one picture



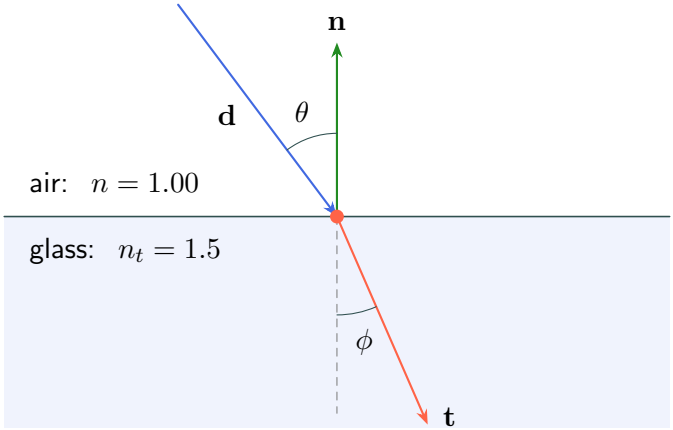
- $\mathbf{d}$ points in; bold $\mathbf{n}$ faces it

Our conventions in one picture



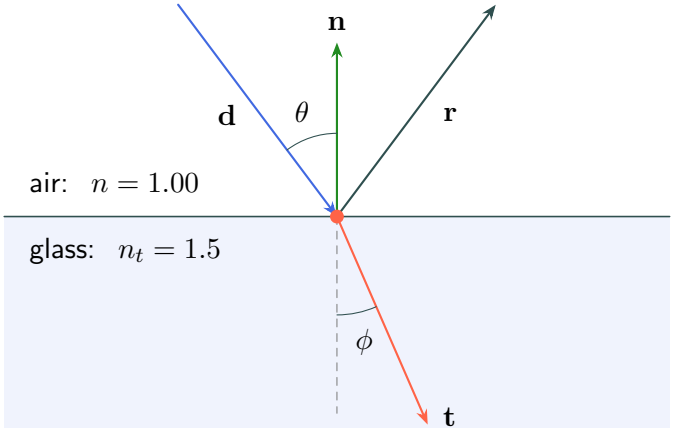
- $\mathbf{d}$ points in; bold $\mathbf{n}$ faces it
- Plain n, n_t: one number per side

Our conventions in one picture



- $\mathbf{d}$ points in; bold $\mathbf{n}$ faces it
- Plain n, n_t: one number per side
- Bold $\mathbf{t}$: the bent direction

Our conventions in one picture

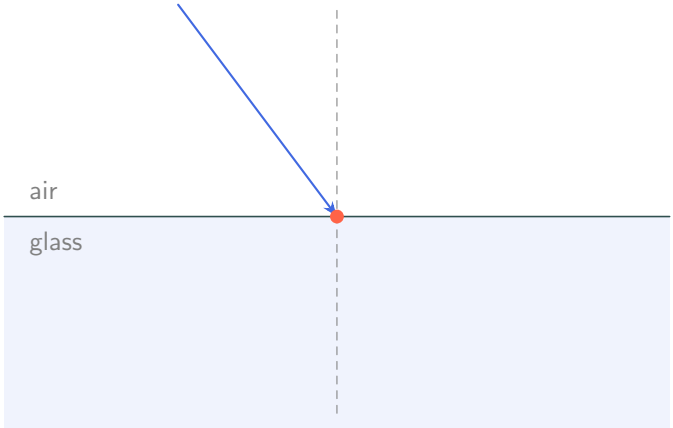


- $\mathbf{d}$ points in; bold $\mathbf{n}$ faces it
- Plain n, n_t: one number per side
- Bold $\mathbf{t}$: the bent direction
- $\mathbf{r}$: lecture 17's mirror direction

Light bends at a glass surface

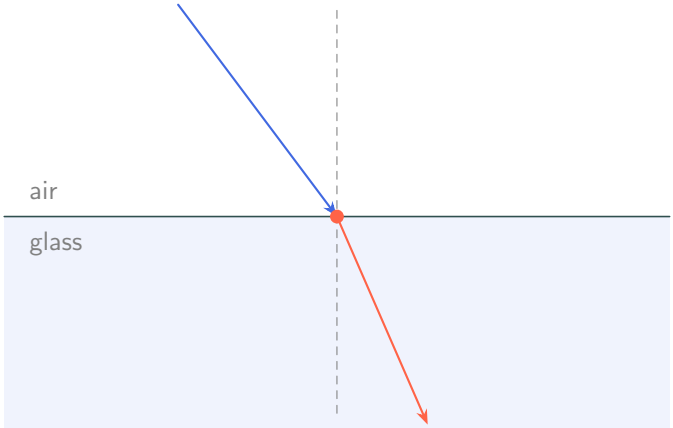
Snell's law: how far the ray turns

A ray meets glass: three questions



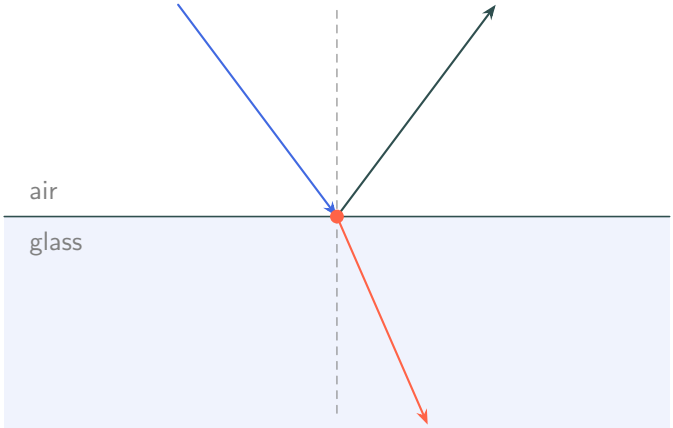
- Clear materials: glass, water, diamond

A ray meets glass: three questions



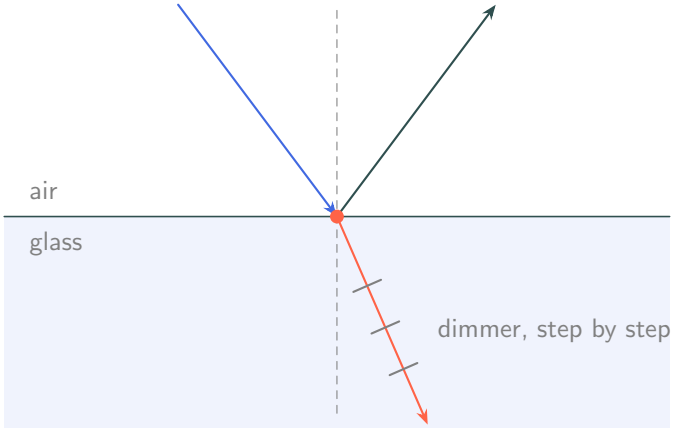
- Clear materials: glass, water, diamond
- Where does the bent ray go?

A ray meets glass: three questions



- Clear materials: glass, water, diamond
- Where does the bent ray go?
- How much light is mirrored?

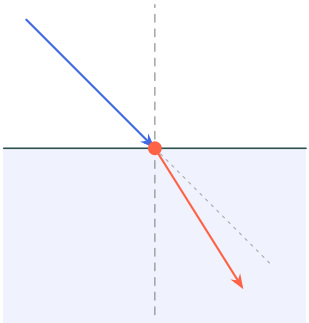
A ray meets glass: three questions



- Clear materials: glass, water, diamond
- Where does the bent ray go?
- How much light is mirrored?
- How much is absorbed inside?

The refractive index n: how strongly light bends

air, $n = 1.00$, above each surface

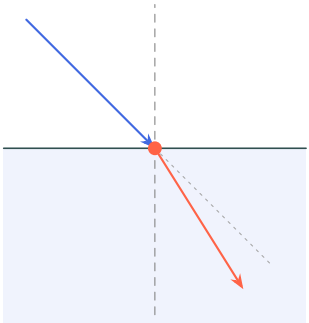


water, $n = 1.33$

- One number per material

The refractive index n: how strongly light bends

air, $n = 1.00$, above each surface



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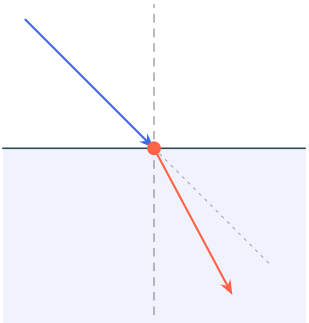
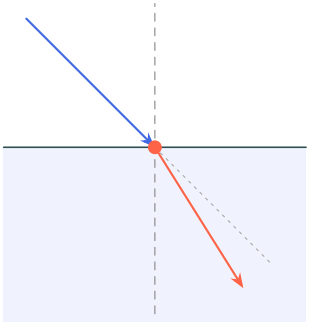


glass, $n = 1.5$

- The same ray arrives

The refractive index n: how strongly light bends

air, $n = 1.00$, above each surface



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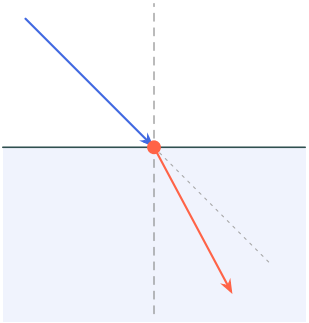
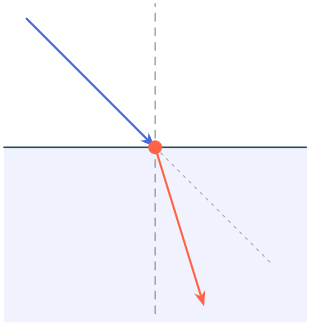


glass, $n = 1.5$

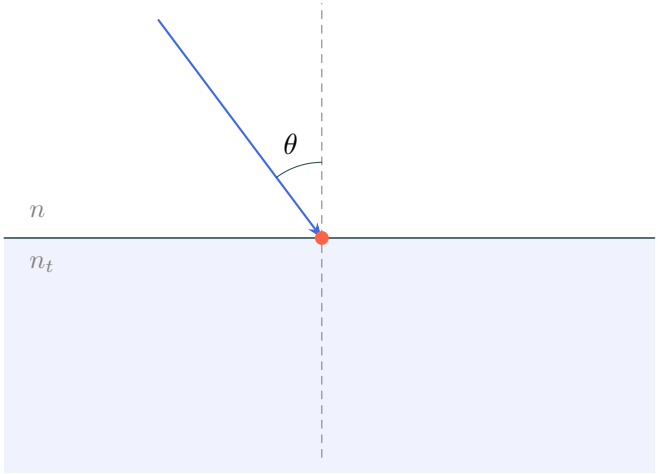
- The same ray arrives



diamond, $n = 2.4$

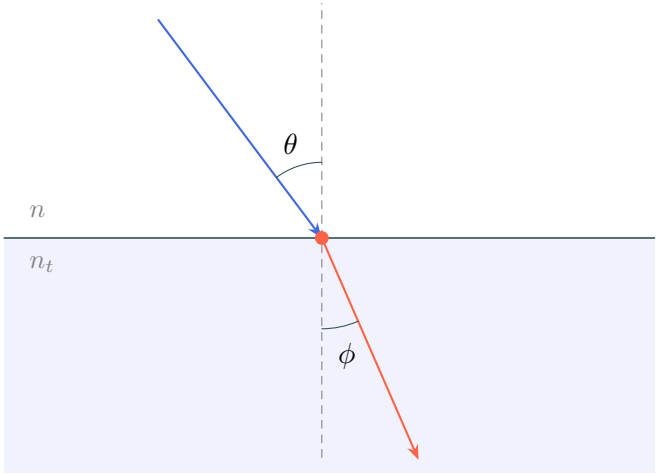
- Bigger n: a stronger bend

Snell's law links the two angles



- Both angles: from the normal

Snell's law links the two angles



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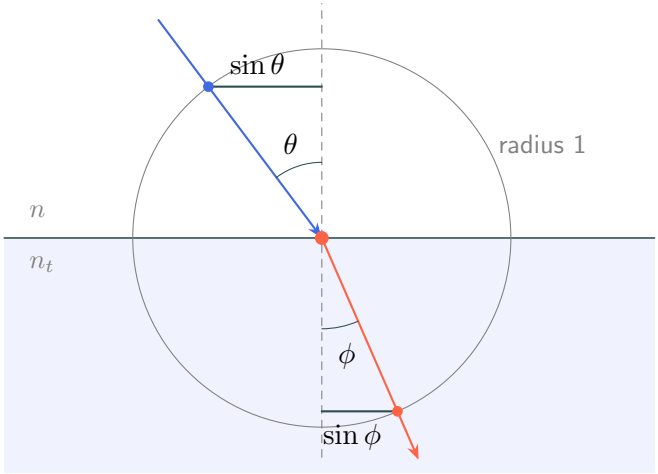
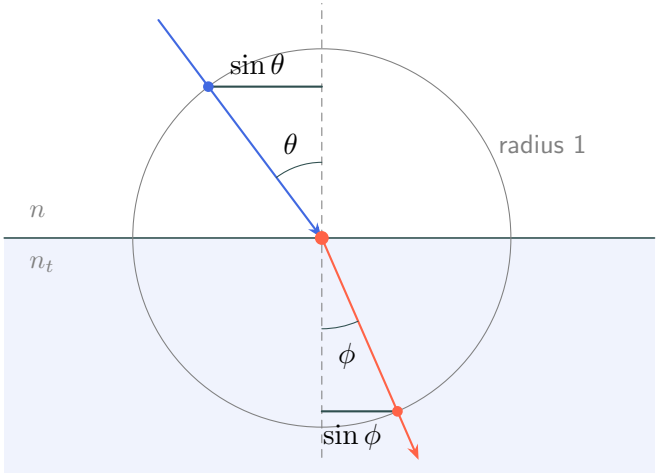


- Both angles: from the normal
- Each sine: a distance from the normal

Snell's law links the two angles



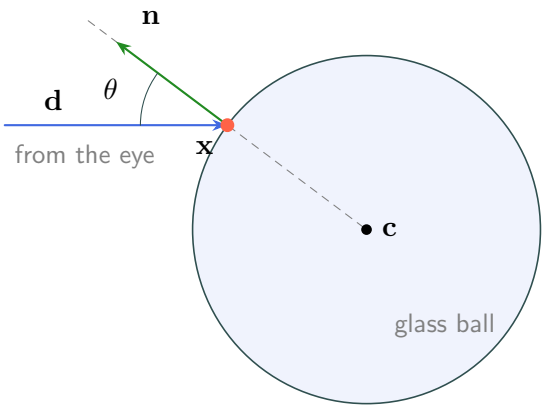
- Both angles: from the normal
- Each sine: a distance from the normal

$$n \sin\theta = n_t \sin\phi$$

Snell's law

- Bigger index, smaller sine

Worked: the angle inside our glass ball

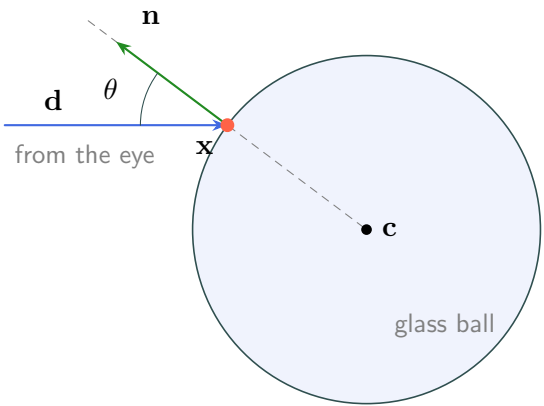


side view, close to $\mathbf{x}$

- Try first: the angle ϕ inside

$$n = 1, \;\; n_t = 1.5, \;\; \theta = 36.87°$$

Worked: the angle inside our glass ball



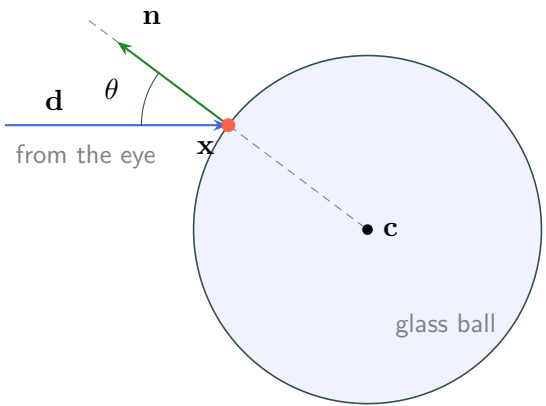
side view, close to $\mathbf{x}$

- Try first: the angle ϕ inside

$$n = 1, \quad n_t = 1.5, \quad \theta = 36.87°$$

$$\sin\theta = \sin 36.87° = 0.6$$

Worked: the angle inside our glass ball



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$$n = 1, \quad n_t = 1.5, \quad \theta = 36.87°$$

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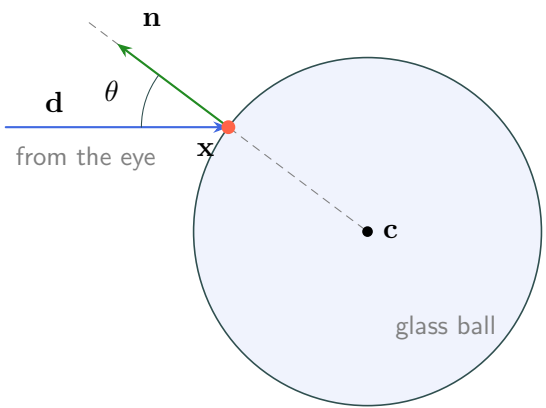
$$1 \times 0.6 = 1.5 \times \sin\phi$$

Worked: the angle inside our glass ball



side view, close to $\mathbf{x}$

- Try first: the angle ϕ inside

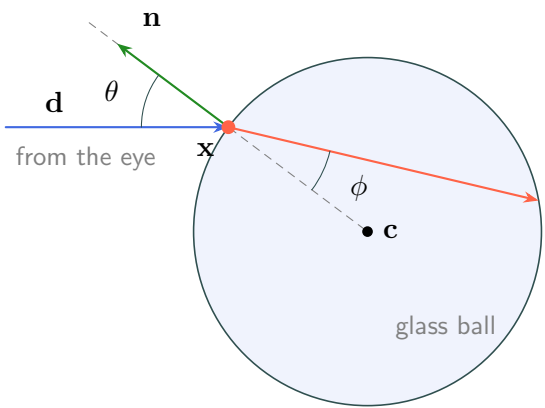
$$n = 1, \quad n_t = 1.5, \quad \theta = 36.87°$$

$$\sin\theta = \sin 36.87° = 0.6$$

$$1 \times 0.6 = 1.5 \times \sin\phi$$

$$\sin\phi = 0.6/1.5 = 0.4$$

Worked: the angle inside our glass ball



side view, close to $\mathbf{x}$

- Try first: the angle ϕ inside

$$n = 1, \quad n_t = 1.5, \quad \theta = 36.87^\circ$$

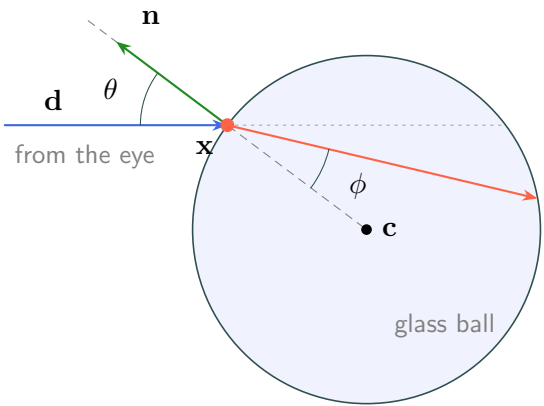
$$\sin\theta = \sin 36.87^\circ = 0.6$$

$$1 \times 0.6 = 1.5 \times \sin\phi$$

$$\sin\phi = 0.6/1.5 = 0.4$$

$$\phi = 23.58^\circ$$

Worked: the angle inside our glass ball



side view, close to $\mathbf{x}$

- Try first: the angle ϕ inside

$$n = 1, \quad n_t = 1.5, \quad \theta = 36.87^\circ$$

$$\sin\theta = \sin 36.87^\circ = 0.6$$

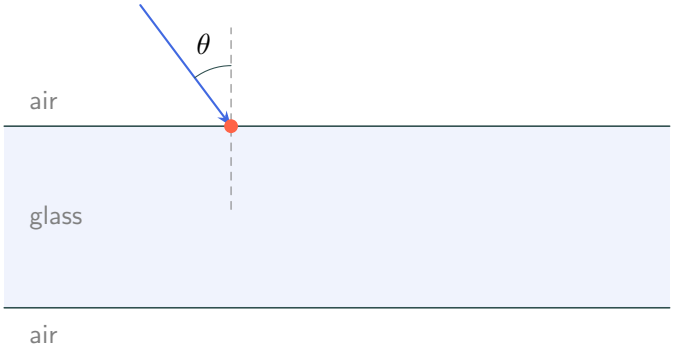
$$1 \times 0.6 = 1.5 \times \sin\phi$$

$$\sin\phi = 0.6/1.5 = 0.4$$

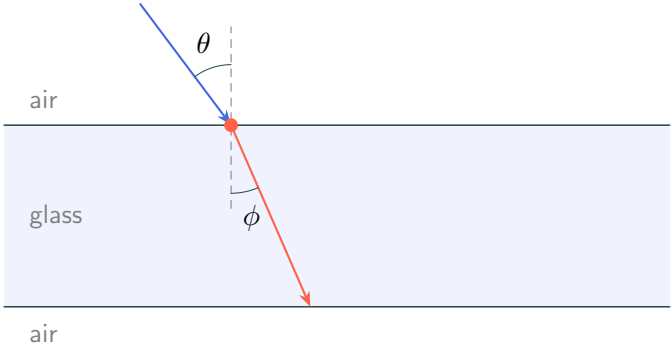
$$\phi = 23.58^\circ$$

- Bent toward the normal

Through a pane: same angle out

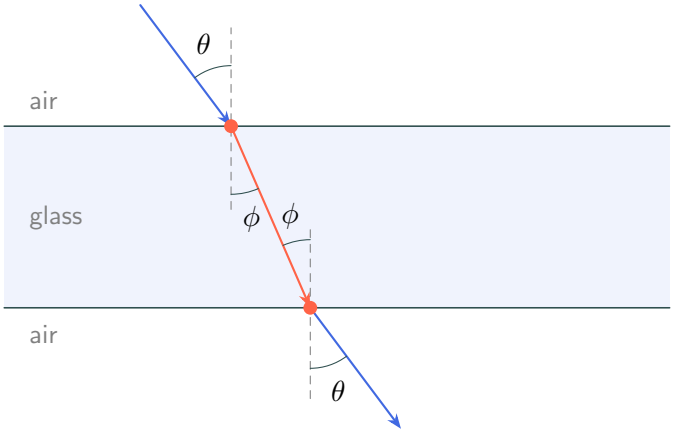


Through a pane: same angle out



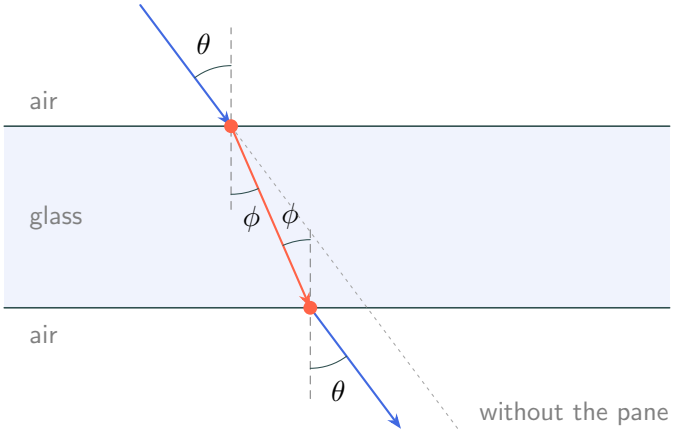
- In: bent toward the normal

Through a pane: same angle out



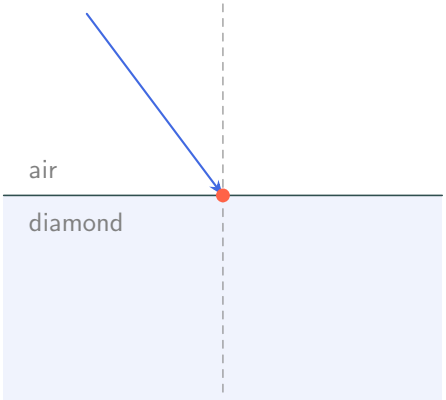
- In: bent toward the normal
- Out: n and n_t swap places
- It bends away, back to θ

Through a pane: same angle out



- In: bent toward the normal
- Out: n and n_t swap places
- It bends away, back to θ
- Same direction, shifted sideways

Quick check 1: into diamond, and out again



Air into diamond: $n = 1$, $n_t = 2.4$, $\sin\theta = 0.6$

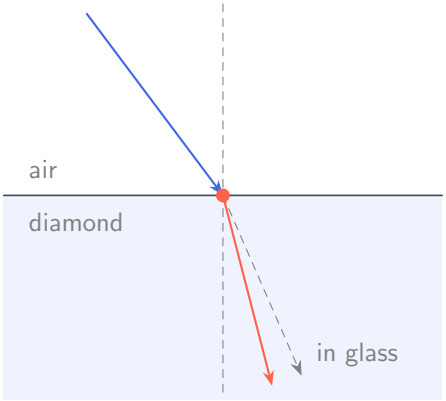
1. $\sin\phi = ?$

2. Glass gave 0.4. Which ray is closer to the normal?

3. The ray turns back, diamond into air, with sine 0.25 inside. Sine in air?

- Three answers from one law

Quick check 1: into diamond, and out again



- Three answers from one law

Air into diamond: $n = 1$, $n_t = 2.4$, $\sin\theta = 0.6$

1. $\sin\phi = ?$
   $\sin\phi = 0.6/2.4 = 0.25$
2. Glass gave 0.4. Which ray is closer to the normal?
   the ray in diamond: $0.25 < 0.4$
3. The ray turns back, diamond into air, with sine 0.25 inside. Sine in air?
   $2.4 \times 0.25 = 0.6$: the angle it came in with

Recap: Snell's law

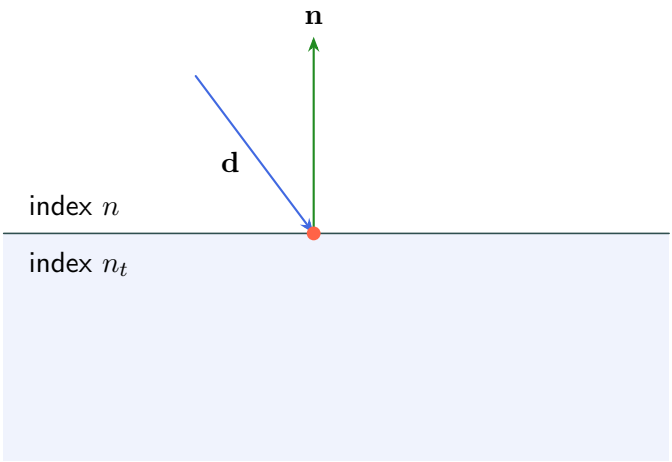
$$n \sin\theta = n_t \sin\phi$$

n, θ: the side the ray comes from n_t, ϕ: the side it goes to

The bent direction, as a vector

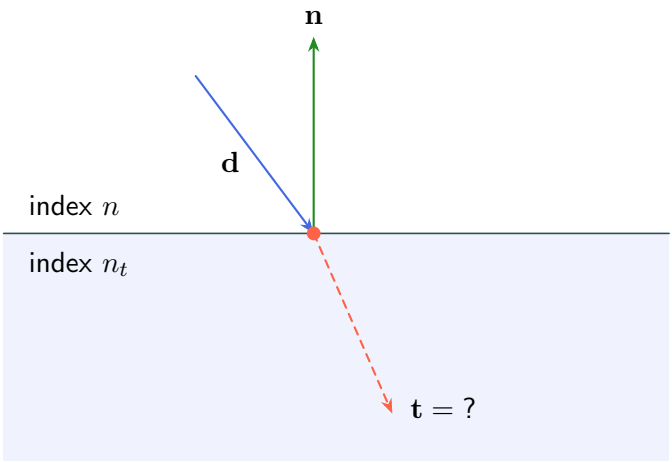
From $\mathbf{d}$ and $\mathbf{n}$ to $\mathbf{t}$, with no angle to measure

Goal: the bent direction $\mathbf{t}$, without angles



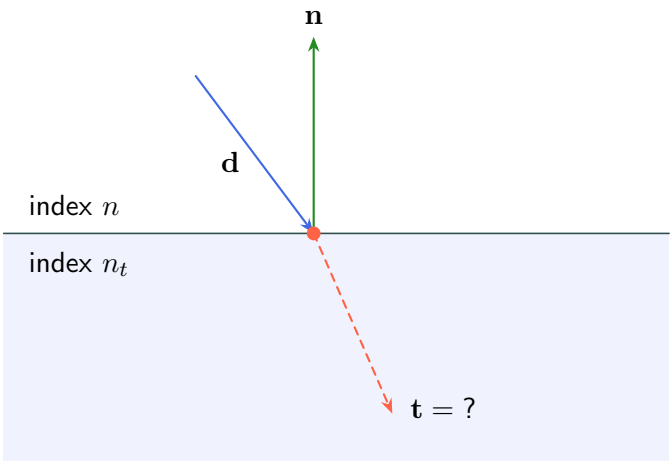
- Known: $\mathbf{d}$, $\mathbf{n}$ and the two indices

Goal: the bent direction $\mathbf{t}$, without angles



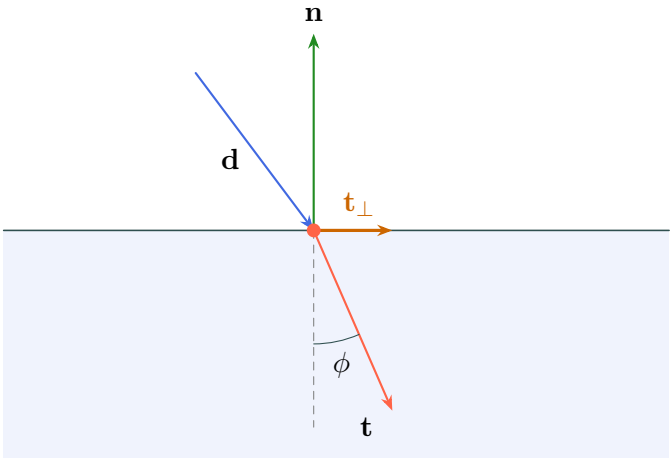
- Known: $\mathbf{d}$, $\mathbf{n}$ and the two indices
- Wanted: the unit arrow $\mathbf{t}$

Goal: the bent direction $\mathbf{t}$, without angles



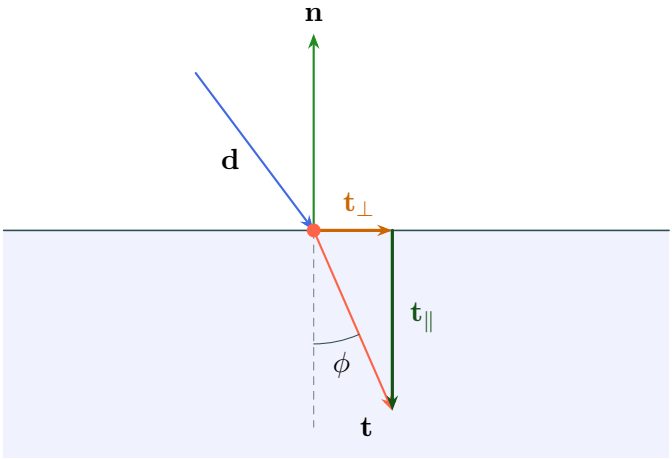
- Known: $\mathbf{d}$, $\mathbf{n}$ and the two indices
- Wanted: the unit arrow $\mathbf{t}$
- Code holds arrows, not angles

Split $\mathbf{t}$ into two components: $\mathbf{t}_\perp$ and $\mathbf{t}_\parallel$



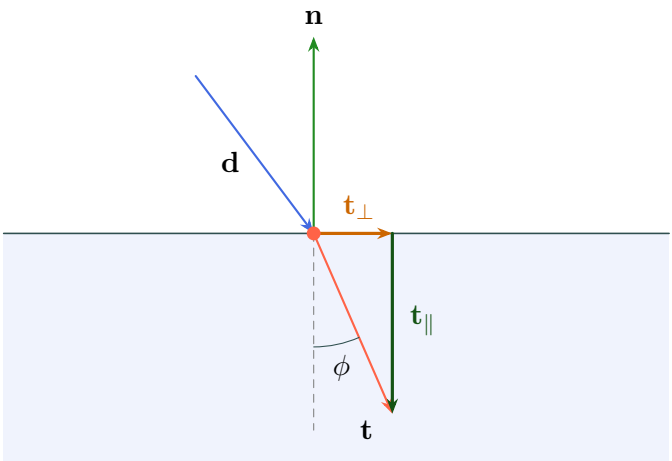
- $\mathbf{t}_\perp$ is perpendicular to $\mathbf{n}$: it lies along the surface.

Split $\mathbf{t}$ into two components: $\mathbf{t}_\perp$ and $\mathbf{t}_\parallel$



- $\mathbf{t}_\perp$ is perpendicular to $\mathbf{n}$: it lies along the surface.
- $\mathbf{t}_\parallel$ is parallel to $\mathbf{n}$: it lies along the normal line.

Split $\mathbf{t}$ into two components: $\mathbf{t}_\perp$ and $\mathbf{t}_\parallel$



- $\mathbf{t}_\perp$ is perpendicular to $\mathbf{n}$: it lies along the surface.
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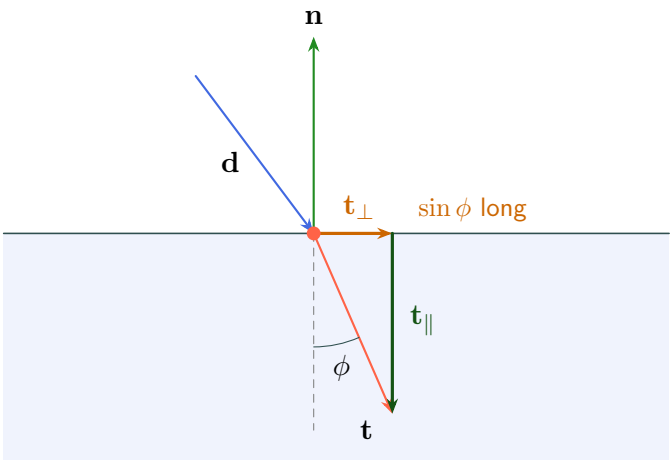
$$\mathbf{t} = \mathbf{t}_\perp + \mathbf{t}_\parallel$$

The subscripts are Ray Tracing in One Weekend's: it writes $\mathbf{R}'_\perp$ and $\mathbf{R}'_\parallel$ for our $\mathbf{t}_\perp$ and $\mathbf{t}_\parallel$

Split $\mathbf{t}$ into two components: $\mathbf{t}_\perp$ and $\mathbf{t}_\parallel$



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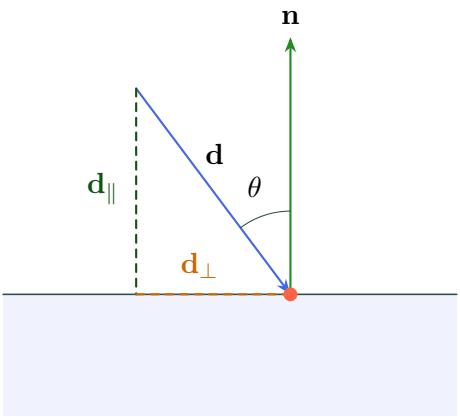
- $\mathbf{t}_\parallel$ is parallel to $\mathbf{n}$: it lies along the normal line.

$$\mathbf{t} = \mathbf{t}_\perp + \mathbf{t}_\parallel$$

- $\mathbf{t}_\perp$ is $\sin\phi$ long: Snell's law gives it. Length 1 then gives $\mathbf{t}_\parallel$.

The subscripts are Ray Tracing in One Weekend's: it writes $\mathbf{R}'_\perp$ and $\mathbf{R}'_\parallel$ for our $\mathbf{t}_\perp$ and $\mathbf{t}_\parallel$

$\mathbf{d}$'s perpendicular component is $\mathbf{d} + \cos\theta\,\mathbf{n}$

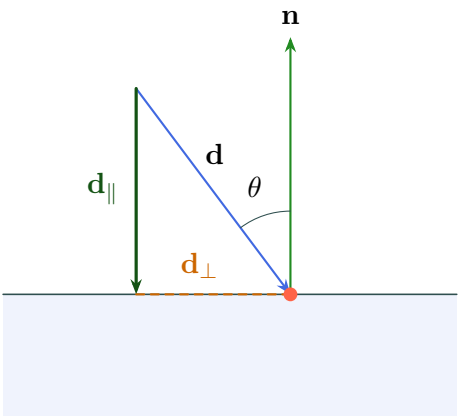


$\perp$: perpendicular to $\mathbf{n}$. $\parallel$: parallel to $\mathbf{n}$.

$$\mathbf{d} = \mathbf{d}_\perp + \mathbf{d}_\parallel$$

split $\mathbf{d}$ the same way

$\mathbf{d}$'s perpendicular component is $\mathbf{d} + \cos\theta\,\mathbf{n}$

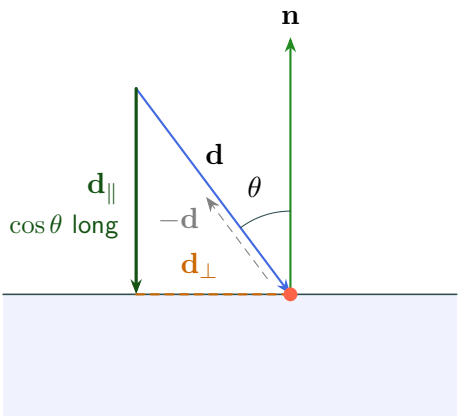


$\perp$: perpendicular to $\mathbf{n}$. $\parallel$: parallel to $\mathbf{n}$.

$\mathbf{d} = \mathbf{d}_\perp + \mathbf{d}_\parallel$ split $\mathbf{d}$ the same way

$\mathbf{d}_\parallel = (\mathbf{d} \cdot \mathbf{n})\,\mathbf{n}$ the part along $\mathbf{n}$ (lecture 17)

$\mathbf{d}$'s perpendicular component is $\mathbf{d} + \cos\theta\,\mathbf{n}$



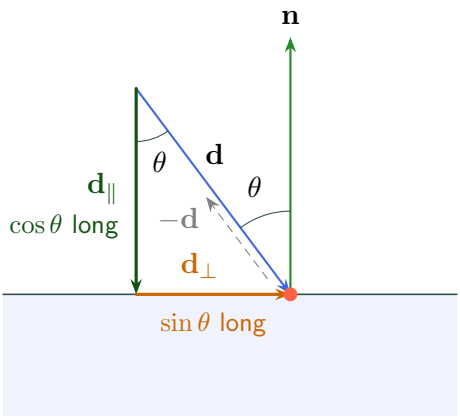
$\mathbf{d} = \mathbf{d}_\perp + \mathbf{d}_\parallel$ — split $\mathbf{d}$ the same way

$\mathbf{d}_\parallel = (\mathbf{d} \cdot \mathbf{n})\,\mathbf{n}$ — the part along $\mathbf{n}$ (lecture 17)

$\mathbf{d} \cdot \mathbf{n} = -\cos\theta$ — θ is between $-\mathbf{d}$ and $\mathbf{n}$, so $(-\mathbf{d}) \cdot \mathbf{n} = \cos\theta$

$\perp$: perpendicular to $\mathbf{n}$. $\parallel$: parallel to $\mathbf{n}$.

$\mathbf{d}$'s perpendicular component is $\mathbf{d} + \cos\theta\,\mathbf{n}$



⊥: perpendicular to $\mathbf{n}$. ∥: parallel to $\mathbf{n}$.

$\mathbf{d} = \mathbf{d}_\perp + \mathbf{d}_\parallel$ — split $\mathbf{d}$ the same way

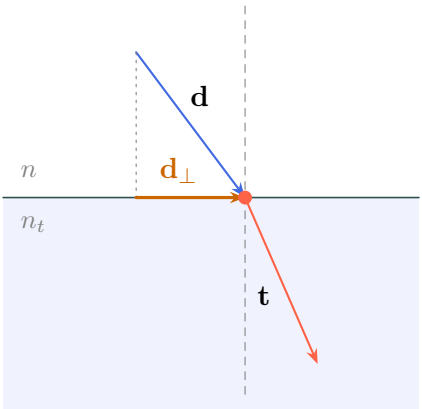
$\mathbf{d}_\parallel = (\mathbf{d} \cdot \mathbf{n})\,\mathbf{n}$ — the part along $\mathbf{n}$ (lecture 17)

$\mathbf{d} \cdot \mathbf{n} = -\cos\theta$ — θ is between $-\mathbf{d}$ and $\mathbf{n}$, so $(-\mathbf{d}) \cdot \mathbf{n} = \cos\theta$

$\mathbf{d}_\perp = \mathbf{d} - \mathbf{d}_\parallel = \mathbf{d} + \cos\theta\ \mathbf{n}$

- $\mathbf{d}_\perp$ has length $\sin\theta$. — right triangle, long side 1: $1 - \cos^2\theta = \sin^2\theta$

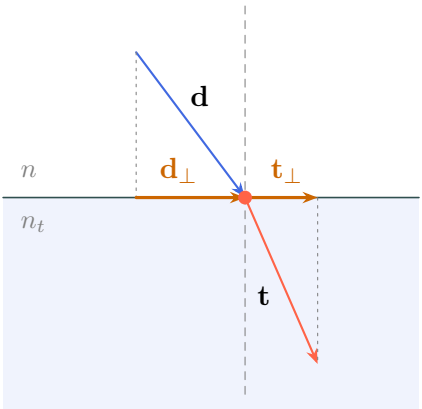
Snell's law scales the perpendicular component



- $\mathbf{d}_\perp$ has length $\sin\theta$.

the last slide

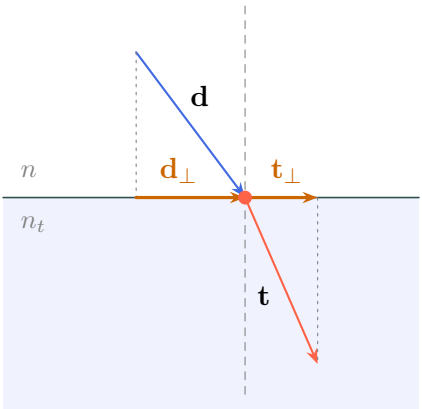
Snell's law scales the perpendicular component



- $\mathbf{d}_\perp$ has length $\sin\theta$. the last slide
- $\mathbf{t}_\perp$ points the same way, with length $\sin\phi$.

$\mathbf{d}$, $\mathbf{n}$ and $\mathbf{t}$ lie in one flat sheet, as in lecture 17.

Snell's law scales the perpendicular component

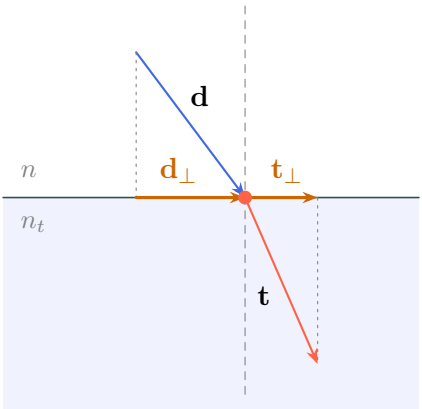


- $\mathbf{d}_\perp$ has length $\sin\theta$. — the last slide
- $\mathbf{t}_\perp$ points the same way, with length $\sin\phi$.

$$\sin\phi = \frac{n}{n_t}\sin\theta$$ — Snell's law

$\mathbf{d}$, $\mathbf{n}$ and $\mathbf{t}$ lie in one flat sheet, as in lecture 17.

Snell's law scales the perpendicular component



- $\mathbf{d}_\perp$ has length $\sin\theta$. — the last slide
- $\mathbf{t}_\perp$ points the same way, with length $\sin\phi$.

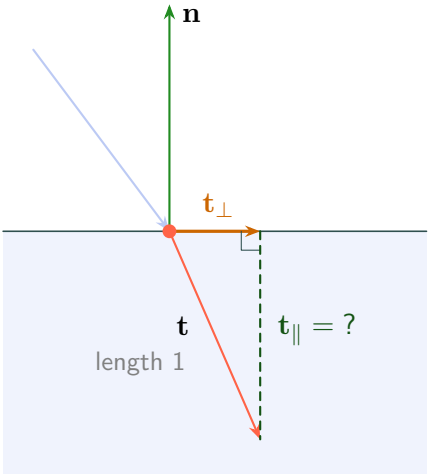
$$\sin\phi = \frac{n}{n_t}\sin\theta$$ — Snell's law

$$\mathbf{t}_\perp = \frac{n}{n_t}\mathbf{d}_\perp = \frac{n}{n_t}(\mathbf{d} + \cos\theta\ \mathbf{n})$$

- From air into glass, $\mathbf{t}_\perp$ is two thirds of $\mathbf{d}_\perp$.

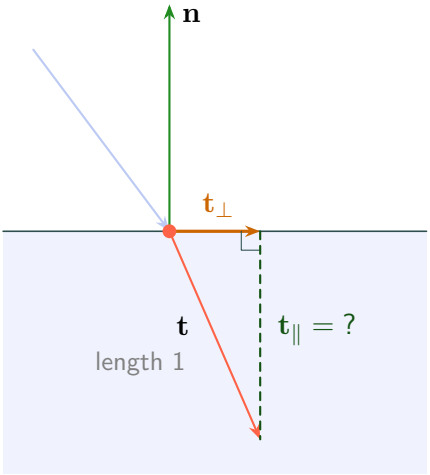
$\mathbf{d}$, $\mathbf{n}$ and $\mathbf{t}$ lie in one flat sheet, as in lecture 17.

Length 1 gives the parallel component



- $\mathbf{t}$ has length 1, and $\mathbf{t}_\perp$ is perpendicular to $\mathbf{t}_\parallel$.

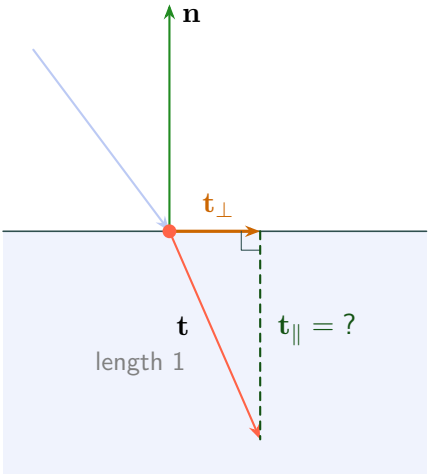
Length 1 gives the parallel component



- $\mathbf{t}$ has length 1, and $\mathbf{t}_\perp$ is perpendicular to $\mathbf{t}_\parallel$.

$$\|\mathbf{t}_\perp\|^2 + \|\mathbf{t}_\parallel\|^2 = 1$$ Pythagoras

Length 1 gives the parallel component

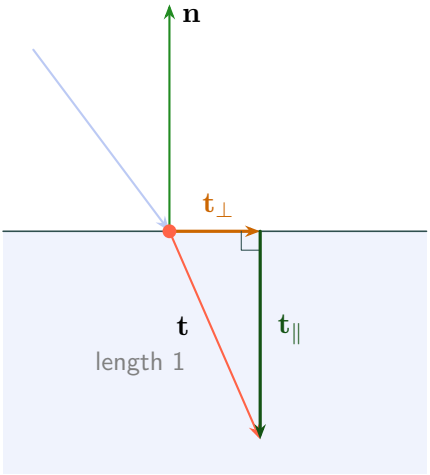


- $\mathbf{t}$ has length 1, and $\mathbf{t}_\perp$ is perpendicular to $\mathbf{t}_\parallel$.

$$\|\mathbf{t}_\perp\|^2 + \|\mathbf{t}_\parallel\|^2 = 1 \qquad \text{Pythagoras}$$

$$\|\mathbf{t}_\parallel\| = \sqrt{1 - \|\mathbf{t}_\perp\|^2} \qquad \text{this length is } \cos\phi$$

Length 1 gives the parallel component



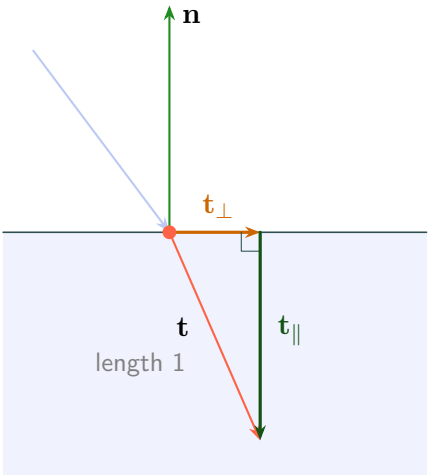
- $\mathbf{t}$ has length 1, and $\mathbf{t}_\perp$ is perpendicular to $\mathbf{t}_\parallel$.

$$\|\mathbf{t}_\perp\|^2 + \|\mathbf{t}_\parallel\|^2 = 1 \qquad \text{Pythagoras}$$

$$\|\mathbf{t}_\parallel\| = \sqrt{1 - \|\mathbf{t}_\perp\|^2} \qquad \text{this length is } \cos\phi$$

$$\mathbf{t}_\parallel = -\sqrt{1 - \|\mathbf{t}_\perp\|^2}\ \mathbf{n} \qquad \text{opposite to } \mathbf{n}$$

Length 1 gives the parallel component



- $\mathbf{t}$ has length 1, and $\mathbf{t}_\perp$ is perpendicular to $\mathbf{t}_\parallel$.

$$\|\mathbf{t}_\perp\|^2 + \|\mathbf{t}_\parallel\|^2 = 1 \qquad \text{Pythagoras}$$

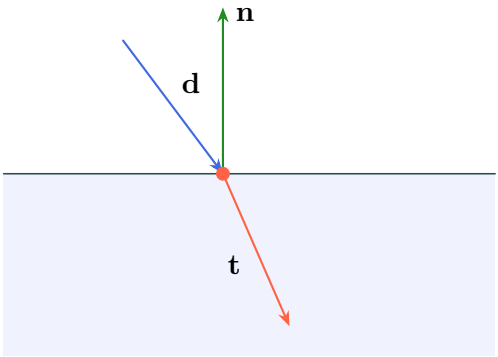
$$\|\mathbf{t}_\parallel\| = \sqrt{1 - \|\mathbf{t}_\perp\|^2} \qquad \text{this length is } \cos\phi$$

$$\mathbf{t}_\parallel = -\sqrt{1 - \|\mathbf{t}_\perp\|^2}\ \mathbf{n} \qquad \text{opposite to } \mathbf{n}$$

$$\|\mathbf{t}_\perp\|^2 = \frac{n^2 \sin^2\theta}{n_t^2} = \frac{n^2\,(1 - \cos^2\theta)}{n_t^2}$$

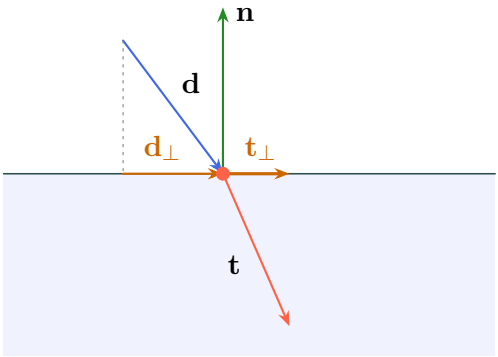
last slide: $\mathbf{t}_\perp = (n/n_t)\,\mathbf{d}_\perp$, and $\mathbf{d}_\perp$ is $\sin\theta$ long
now only $\cos\theta$ is left: one dot product

The refraction formula has two parts



$\mathbf{t} = \mathbf{t}_\perp + \mathbf{t}_\parallel$ the two components, added

The refraction formula has two parts

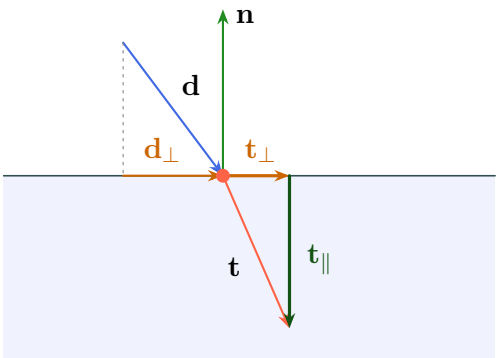


$\mathbf{t} = \mathbf{t}_\perp + \mathbf{t}_\parallel$ the two components, added

- $\mathbf{t}_\perp = \dfrac{n\,(\mathbf{d} + \mathbf{n}\cos\theta)}{n_t}$

  Snell's law: $(n/n_t)\,\mathbf{d}_\perp$, and $\mathbf{d}_\perp = \mathbf{d} + \cos\theta\ \mathbf{n}$

The refraction formula has two parts



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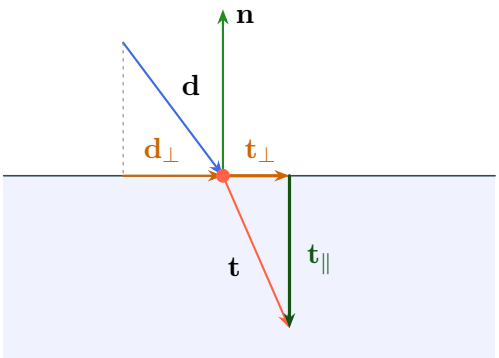
  Snell's law: $(n/n_t)\,\mathbf{d}_\perp$, and $\mathbf{d}_\perp = \mathbf{d} + \cos\theta\ \mathbf{n}$

- $\mathbf{t}_\parallel = -\,\mathbf{n}\sqrt{1 - \dfrac{n^2\,(1-\cos^2\theta)}{n_t^2}}$

  length 1: $-\sqrt{1 - \|\mathbf{t}_\perp\|^2}\ \mathbf{n}$, opposite to $\mathbf{n}$

The refraction formula has two parts



$\mathbf{t} = \mathbf{t}_\perp + \mathbf{t}_\parallel$ the two components, added

- $\mathbf{t}_\perp = \dfrac{n\,(\mathbf{d} + \mathbf{n}\cos\theta)}{n_t}$

  Snell's law: $(n/n_t)\,\mathbf{d}_\perp$, and $\mathbf{d}_\perp = \mathbf{d} + \cos\theta\ \mathbf{n}$

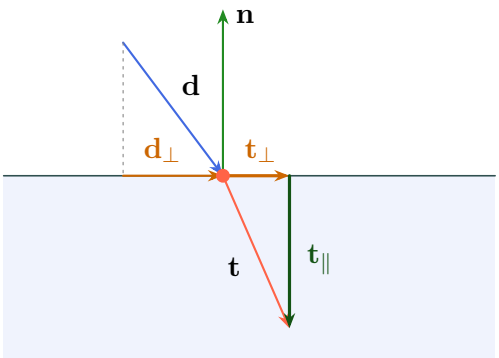
- $\mathbf{t}_\parallel = -\,\mathbf{n}\sqrt{1 - \dfrac{n^2\,(1-\cos^2\theta)}{n_t^2}}$

  length 1: $-\sqrt{1 - \|\mathbf{t}_\perp\|^2}\ \mathbf{n}$, opposite to $\mathbf{n}$

- Put $\cos\theta = -\mathbf{d}\cdot\mathbf{n}$ into both parts: $+\,\mathbf{n}\cos\theta$ becomes $-\,\mathbf{n}\,(\mathbf{d}\cdot\mathbf{n})$, and $\cos^2\theta$ becomes $(\mathbf{d}\cdot\mathbf{n})^2$.

$$\mathbf{t} = \frac{n\,\big(\mathbf{d} - \mathbf{n}\,(\mathbf{d}\cdot\mathbf{n})\big)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\,\big(1 - (\mathbf{d}\cdot\mathbf{n})^2\big)}{n_t^2}}$$

The refraction formula has two parts



$\mathbf{t} = \mathbf{t}_\perp + \mathbf{t}_\parallel$ the two components, added

- $\mathbf{t}_\perp = \dfrac{n\,(\mathbf{d} + \mathbf{n}\cos\theta)}{n_t}$

  Snell's law: $(n/n_t)\,\mathbf{d}_\perp$, and $\mathbf{d}_\perp = \mathbf{d} + \cos\theta\ \mathbf{n}$

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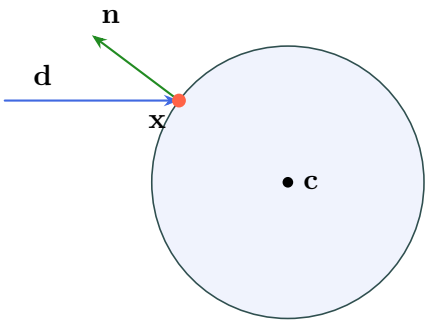
$$\mathbf{t} = \frac{n\,\big(\mathbf{d} - \mathbf{n}\,(\mathbf{d}\cdot\mathbf{n})\big)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\,\big(1 - (\mathbf{d}\cdot\mathbf{n})^2\big)}{n_t^2}}$$

- It works going in and coming out.

Worked: the bent direction at our hit

side view, close to the hit



$\mathbf{d} = (0, 0, -1), \quad \mathbf{n} = (0, 0.6, 0.8)$

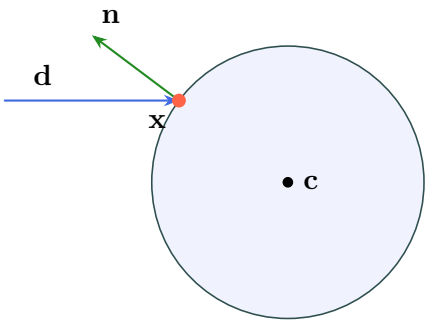
$n = 1, \quad n_t = 1.5$

- Try first: the bent direction $\mathbf{t}$

$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}(\mathbf{d}\cdot\mathbf{n})\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - (\mathbf{d}\cdot\mathbf{n})^2\right)}{n_t^2}}$$

Worked: the bent direction at our hit

side view, close to the hit



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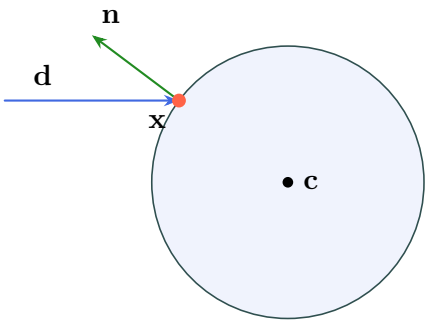
$$\mathbf{d}\cdot\mathbf{n} = (-1)(0.8) = -0.8$$

Worked: the bent direction at our hit

side view, close to the hit



$\mathbf{d} = (0, 0, -1), \quad \mathbf{n} = (0, 0.6, 0.8)$

$n = 1, \quad n_t = 1.5$

$\perp$: along the ball's surface at the hit

- Try first: the bent direction $\mathbf{t}$

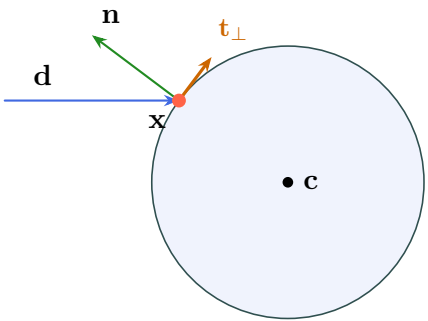
$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}(\mathbf{d}\cdot\mathbf{n})\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - (\mathbf{d}\cdot\mathbf{n})^2\right)}{n_t^2}}$$

$$\mathbf{d}\cdot\mathbf{n} = (-1)(0.8) = -0.8$$

$$\mathbf{d}_\perp = \mathbf{d} + 0.8\,\mathbf{n} = (0, 0.48, -0.36)$$

Worked: the bent direction at our hit

side view, close to the hit



$\mathbf{d} = (0, 0, -1), \quad \mathbf{n} = (0, 0.6, 0.8)$

$n = 1, \quad n_t = 1.5$

$\perp$: along the ball's surface at the hit

- Try first: the bent direction $\mathbf{t}$

$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}(\mathbf{d}\cdot\mathbf{n})\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - (\mathbf{d}\cdot\mathbf{n})^2\right)}{n_t^2}}$$

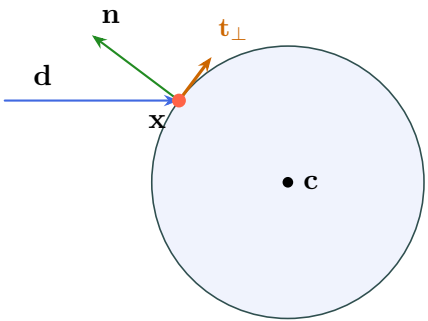
$$\mathbf{d}\cdot\mathbf{n} = (-1)(0.8) = -0.8$$

$$\mathbf{d}_\perp = \mathbf{d} + 0.8\,\mathbf{n} = (0, 0.48, -0.36)$$

$$\mathbf{t}_\perp = \tfrac{1}{1.5}\,\mathbf{d}_\perp = (0, 0.32, -0.24)$$

Worked: the bent direction at our hit

side view, close to the hit



$\mathbf{d} = (0, 0, -1), \quad \mathbf{n} = (0, 0.6, 0.8)$

$n = 1, \quad n_t = 1.5$

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- Try first: the bent direction $\mathbf{t}$

$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}\left(\mathbf{d}\cdot\mathbf{n}\right)\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - \left(\mathbf{d}\cdot\mathbf{n}\right)^2\right)}{n_t^2}}$$

$$\mathbf{d}\cdot\mathbf{n} = (-1)(0.8) = -0.8$$

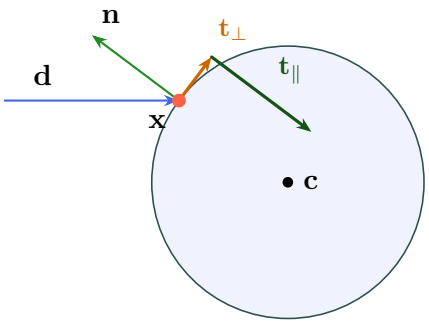
$$\mathbf{d}_\perp = \mathbf{d} + 0.8\,\mathbf{n} = (0, 0.48, -0.36)$$

$$\mathbf{t}_\perp = \tfrac{1}{1.5}\,\mathbf{d}_\perp = (0, 0.32, -0.24)$$

root: $\sqrt{1 - \frac{1-0.64}{2.25}} = \sqrt{0.84} = 0.9165$

Worked: the bent direction at our hit

side view, close to the hit



$\mathbf{d} = (0, 0, -1), \quad \mathbf{n} = (0, 0.6, 0.8)$

$n = 1, \quad n_t = 1.5$

$\perp$: along the ball's surface at the hit

- Try first: the bent direction $\mathbf{t}$

$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}\left(\mathbf{d}\cdot\mathbf{n}\right)\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - \left(\mathbf{d}\cdot\mathbf{n}\right)^2\right)}{n_t^2}}$$

$\mathbf{d}\cdot\mathbf{n} = (-1)(0.8) = -0.8$

$\mathbf{d}_\perp = \mathbf{d} + 0.8\,\mathbf{n} = (0, 0.48, -0.36)$

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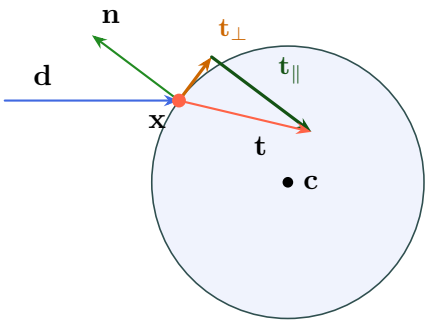
root: $\sqrt{1 - \frac{1-0.64}{2.25}} = \sqrt{0.84} = 0.9165$

$\mathbf{t}_\parallel = -0.9165\,\mathbf{n} = (0, -0.5499, -0.7332)$

Worked: the bent direction at our hit

side view, close to the hit



$\mathbf{d} = (0, 0, -1), \quad \mathbf{n} = (0, 0.6, 0.8)$

$n = 1, \quad n_t = 1.5$

$\perp$: along the ball's surface at the hit

- Try first: the bent direction $\mathbf{t}$

$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}\left(\mathbf{d}\cdot\mathbf{n}\right)\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - \left(\mathbf{d}\cdot\mathbf{n}\right)^2\right)}{n_t^2}}$$

$\mathbf{d}\cdot\mathbf{n} = (-1)(0.8) = -0.8$

$\mathbf{d}_\perp = \mathbf{d} + 0.8\,\mathbf{n} = (0, 0.48, -0.36)$

$\mathbf{t}_\perp = \frac{1}{1.5}\,\mathbf{d}_\perp = (0, 0.32, -0.24)$

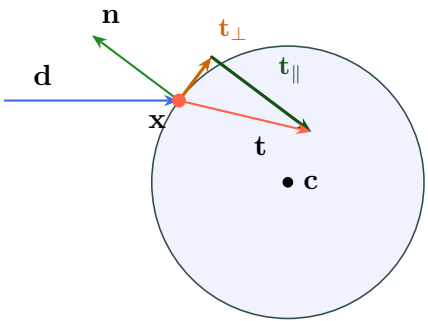
root: $\sqrt{1 - \frac{1-0.64}{2.25}} = \sqrt{0.84} = 0.9165$

$\mathbf{t}_\parallel = -0.9165\,\mathbf{n} = (0, -0.5499, -0.7332)$

$\mathbf{t} = \mathbf{t}_\perp + \mathbf{t}_\parallel = (0, -0.2299, -0.9732)$

Worked: the bent direction at our hit

side view, close to the hit



$\mathbf{d} = (0, 0, -1), \quad \mathbf{n} = (0, 0.6, 0.8)$

$n = 1, \quad n_t = 1.5$

$\perp$: along the ball's surface at the hit

- Try first: the bent direction $\mathbf{t}$

$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}\left(\mathbf{d}\cdot\mathbf{n}\right)\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - \left(\mathbf{d}\cdot\mathbf{n}\right)^2\right)}{n_t^2}}$$

$\mathbf{d}\cdot\mathbf{n} = (-1)(0.8) = -0.8$

$\mathbf{d}_\perp = \mathbf{d} + 0.8\,\mathbf{n} = (0, 0.48, -0.36)$

$\mathbf{t}_\perp = \frac{1}{1.5}\,\mathbf{d}_\perp = (0, 0.32, -0.24)$

root: $\sqrt{1 - \frac{1-0.64}{2.25}} = \sqrt{0.84} = 0.9165$

$\mathbf{t}_\parallel = -0.9165\,\mathbf{n} = (0, -0.5499, -0.7332)$

$\mathbf{t} = \mathbf{t}_\perp + \mathbf{t}_\parallel = (0, -0.2299, -0.9732)$

- Check: $\|\mathbf{t}_\perp\| = 0.4 = \sin\phi$

Recap: the refraction formula

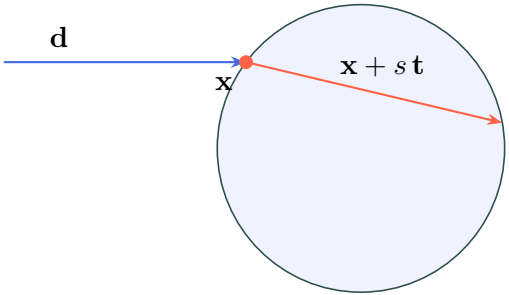
$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}\left(\mathbf{d}\cdot\mathbf{n}\right)\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - (\mathbf{d}\cdot\mathbf{n})^2\right)}{n_t^2}}$$

$\mathbf{d}$, bold $\mathbf{n}$: length 1, $\mathbf{n}$ faces the ray plain n, n_t: the two indices

Leaving the glass

The same formula, and sometimes no way out

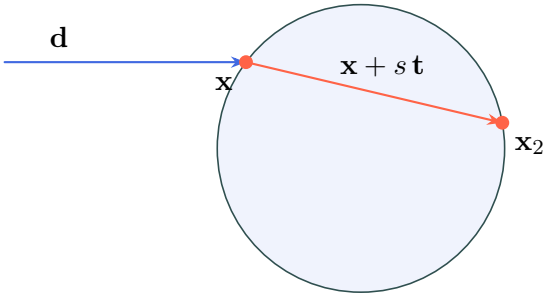
Leaving: a hit from inside, indices swap



side view

- The bent ray: a new ray

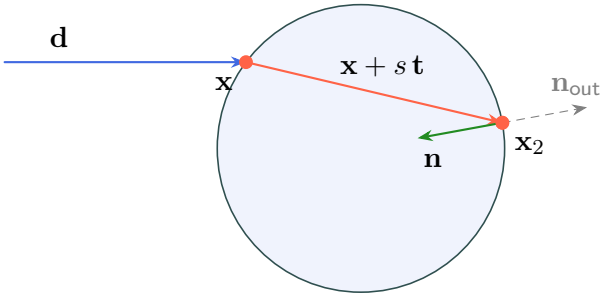
Leaving: a hit from inside, indices swap



side view

- The bent ray: a new ray
- It hits the ball from inside

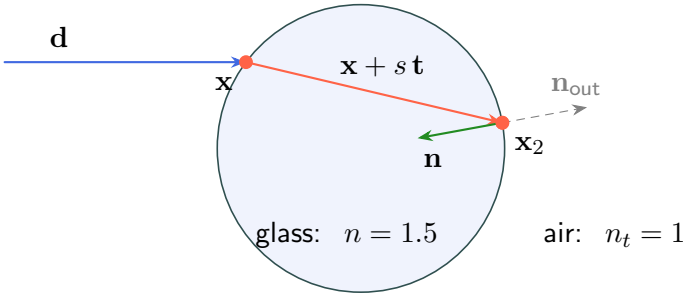
Leaving: a hit from inside, indices swap



side view

- The bent ray: a new ray
- It hits the ball from inside
- Stored $\mathbf{n}$ faces the ray: $-\mathbf{n}_{\text{out}}$

Leaving: a hit from inside, indices swap



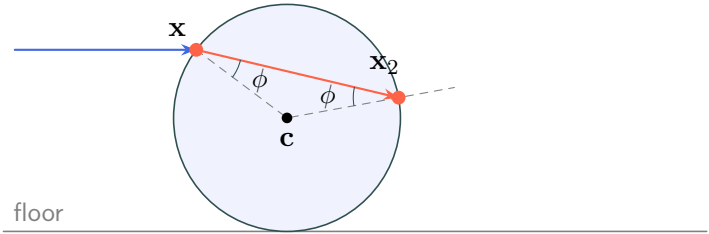
side view

- The bent ray: a new ray
- It hits the ball from inside
- Stored $\mathbf{n}$ faces the ray: $-\mathbf{n}_{\text{out}}$
- Now $n = 1.5$ and $n_t = 1$

Worked: the ray leaves the ball

- Try first: the sine of the angle outside, at $\mathbf{x}_2$

inside, at $\mathbf{x}_2$: $\sin\phi = 0.4$; $n = 1.5$, $n_t = 1$ two radii, so the angle at $\mathbf{x}_2$ is ϕ again



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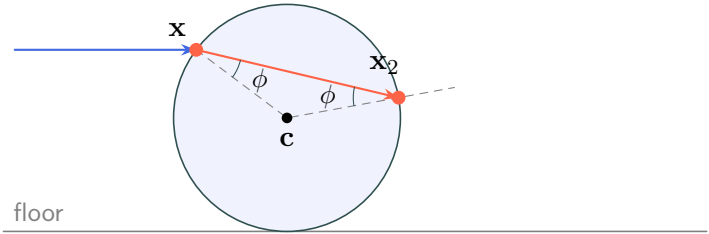
$1.5 \times 0.4 = 1 \times \sin(\text{outside})$ — Snell's law, the indices swapped



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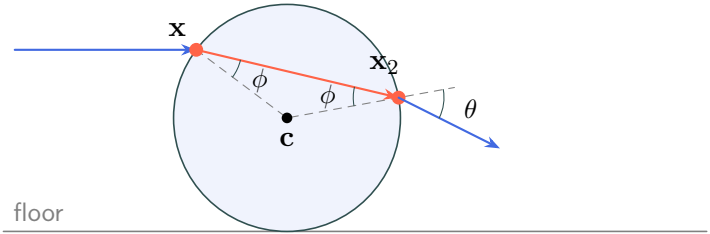
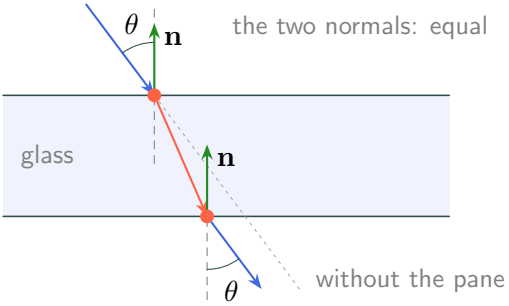
$1.5 \times 0.4 = 1 \times \sin(\text{outside})$ — Snell's law, the indices swapped

$\sin(\text{outside}) = 0.6$: $36.87°$, our θ



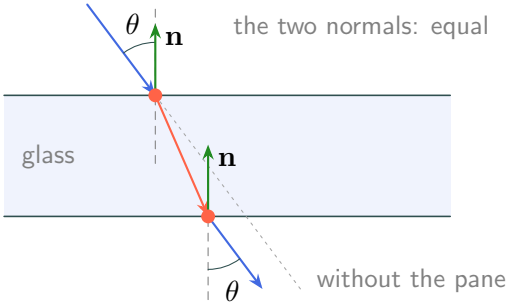
- In at θ, out at θ

Out at θ again, but in a new direction: the two normals differ

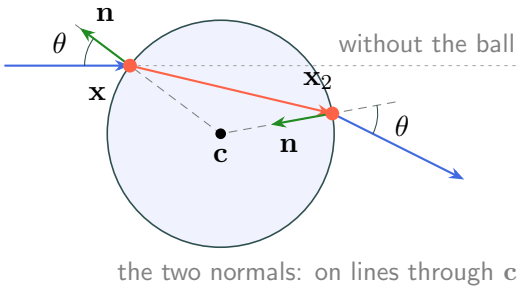


- Pane: both faces have the same normal.
  Out at θ is the old direction.

Out at θ again, but in a new direction: the two normals differ

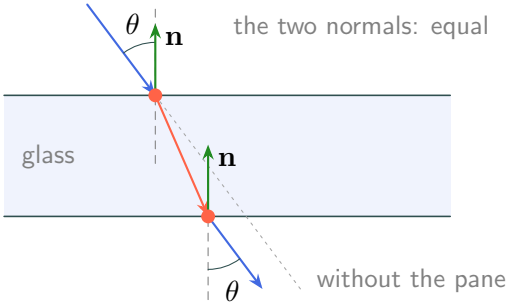


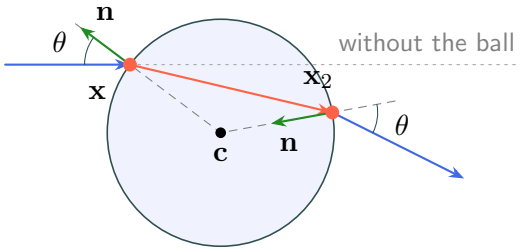
- Pane: both faces have the same normal. Out at θ is the old direction.



- Ball: the normal at $\mathbf{x}_2$ points another way. Out at θ is a new direction.

Out at θ again, but in a new direction: the two normals differ





- Pane: both faces have the same normal. Out at θ is the old direction.
- Ball: the normal at $\mathbf{x}_2$ points another way. Out at θ is a new direction.

Each face turns the ray by $\theta - \phi = 36.87° - 23.58° = 13.29°$.
The pane turns it back; the ball turns it further: $26.58°$ in all.

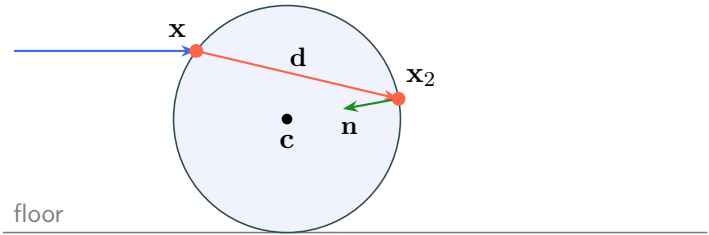
The middle pixel shows the floor

- Which way does the ray go outside? The refraction formula, at $\mathbf{x}_2$:

$\mathbf{d} = (0,\ -0.2299,\ -0.9732)$ the ray inside the ball

$\mathbf{n} = (0,\ -0.1786,\ 0.9839)$ the normal at $\mathbf{x}_2$, turned to face the ray

$n = 1.5, \quad n_t = 1$ from glass into air



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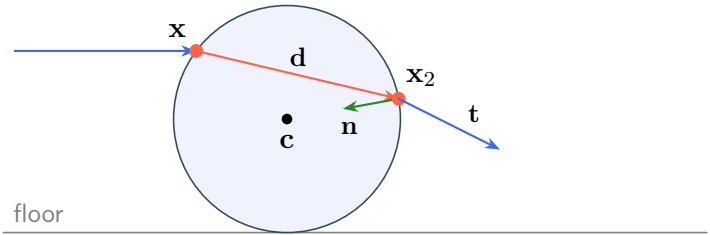
$\mathbf{n} = (0,\ -0.1786,\ 0.9839)$	the normal at $\mathbf{x}_2$, turned to face the ray
$n = 1.5, \quad n_t = 1$	from glass into air
$\mathbf{t} = (0,\ -0.4475,\ -0.8943)$	under the root: $1 - 2.25 \times 0.16 = 0.64$



The middle pixel shows the floor

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$\mathbf{d} = (0,\ -0.2299,\ -0.9732)$	the ray inside the ball
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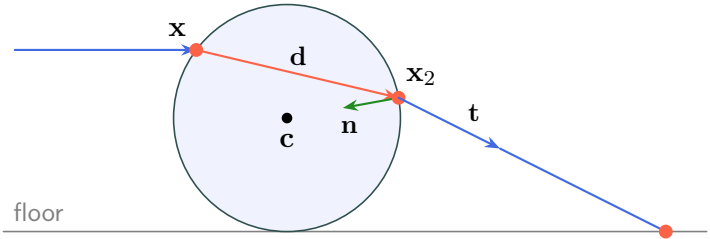

- It meets the floor at $(0,\ -1.6,\ -7.14)$.

The middle pixel shows the floor

- Which way does the ray go outside? The refraction formula, at $\mathbf{x}_2$:

$\mathbf{d} = (0,\ -0.2299,\ -0.9732)$ the ray inside the ball

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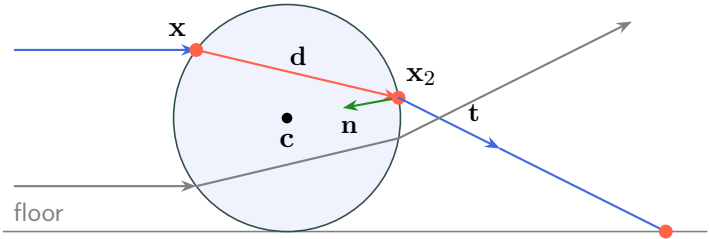
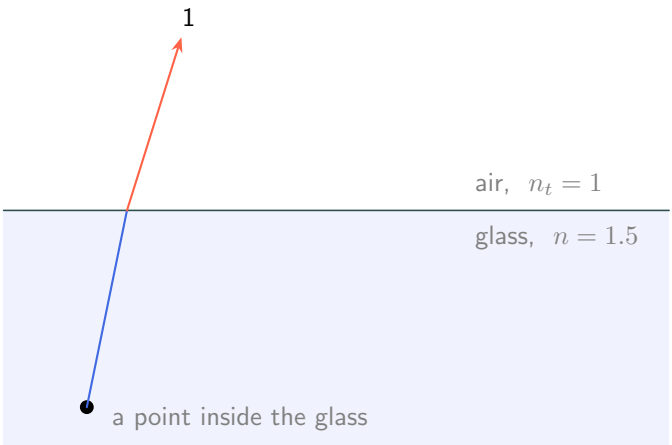


- It meets the floor at $(0,\ -1.6,\ -7.14)$.
- Upper rays are bent down, lower rays up: the scene appears upside down.

Leaving glass, the sine can pass 1

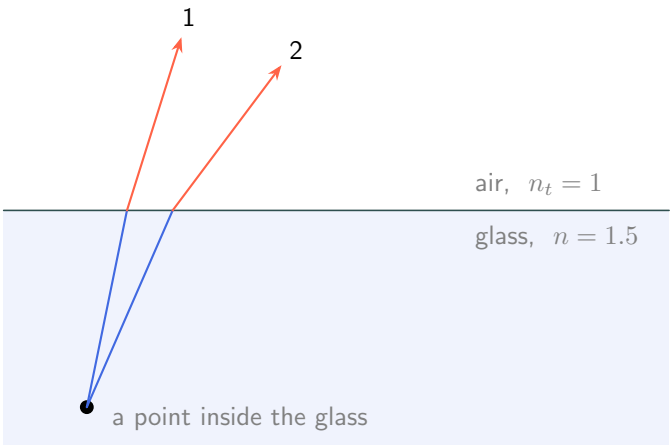


- Out of glass, into air

$$\sin\phi = 1.5\,\sin\theta$$

$$1: \quad 1.5 \times 0.2 = 0.3$$

Leaving glass, the sine can pass 1



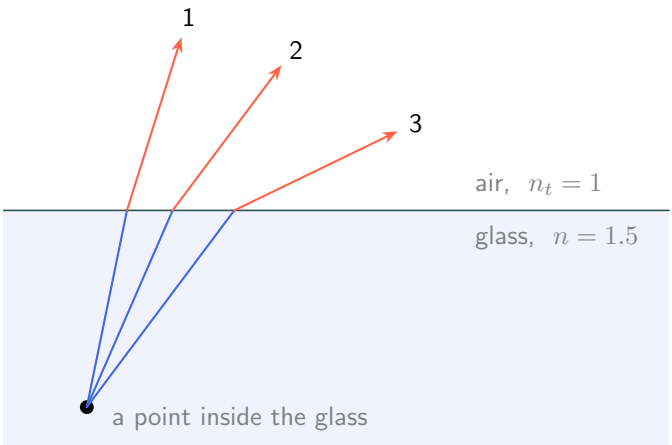
- Out of glass, into air

$$\sin\phi = 1.5\,\sin\theta$$

1: $1.5 \times 0.2 = 0.3$

2: $1.5 \times 0.4 = 0.6$

Leaving glass, the sine can pass 1



- Out of glass, into air

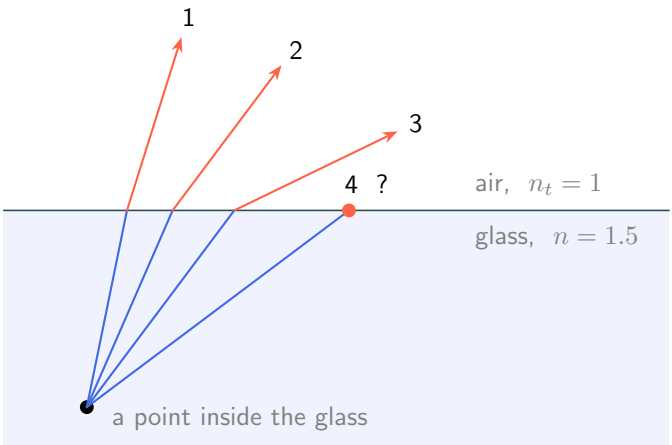
$$\sin\phi = 1.5\,\sin\theta$$

1: $1.5 \times 0.2 = 0.3$

2: $1.5 \times 0.4 = 0.6$

3: $1.5 \times 0.6 = 0.9$

Leaving glass, the sine can pass 1



- Out of glass, into air

$$\sin\phi = 1.5 \sin\theta$$

1: $1.5 \times 0.2 = 0.3$

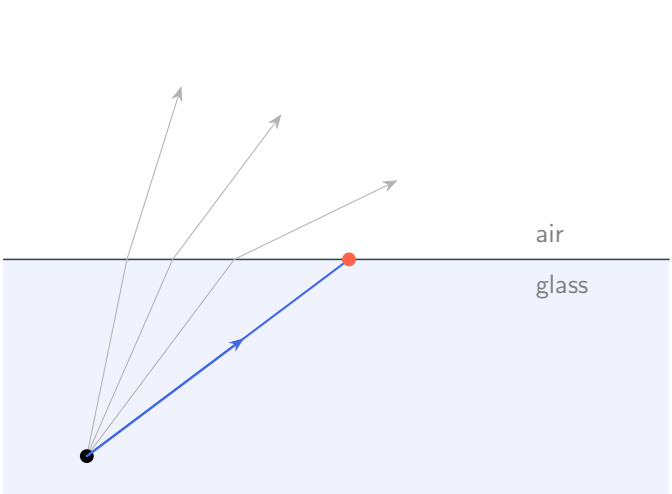
2: $1.5 \times 0.4 = 0.6$

3: $1.5 \times 0.6 = 0.9$

4: $1.5 \times 0.8 = 1.2$

- No angle has sine 1.2

No way out: total internal reflection



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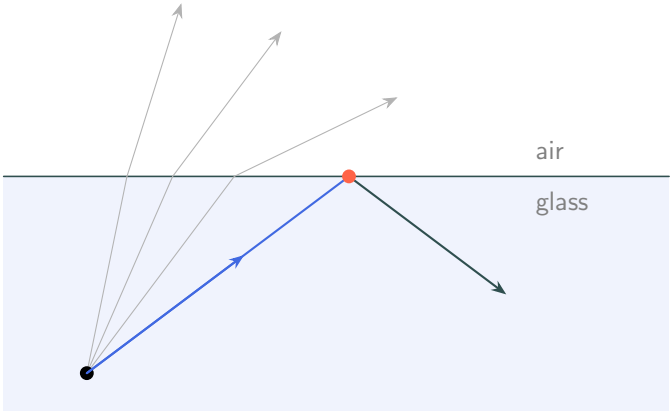
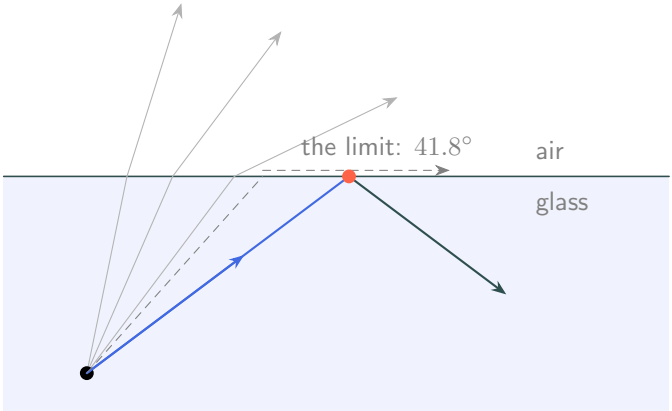


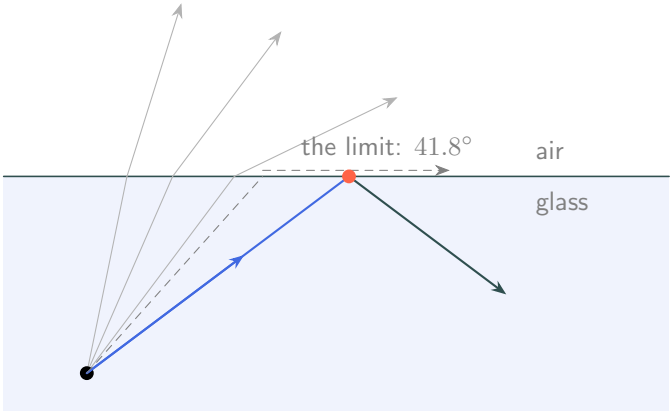
- No bent ray: all light is mirrored
- Name: total internal reflection

No way out: total internal reflection



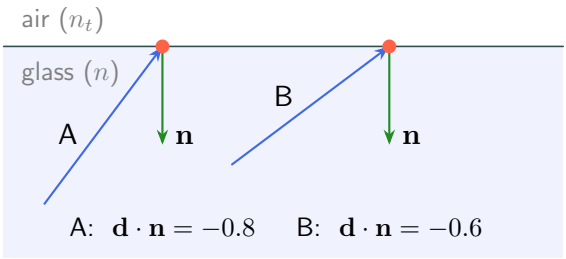
- No bent ray: all light is mirrored
- Name: total internal reflection
- Limit for glass: $\sin\theta = 1/1.5$

No way out: total internal reflection



- No bent ray: all light is mirrored
- Name: total internal reflection
- Limit for glass: $\sin\theta = 1/1.5$
- Formula: a negative number under the root

Worked: can these two rays leave?



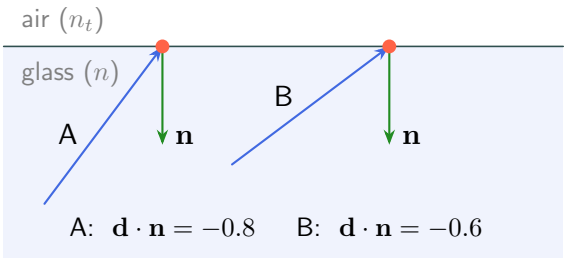
- Try first: which ray leaves the glass?

  the number under the formula's root:

  $$1 - \frac{n^2}{n_t^2}\left(1 - (\mathbf{d} \cdot \mathbf{n})^2\right)$$

  $n = 1.5$, $n_t = 1$: $n^2/n_t^2 = 2.25$

Worked: can these two rays leave?



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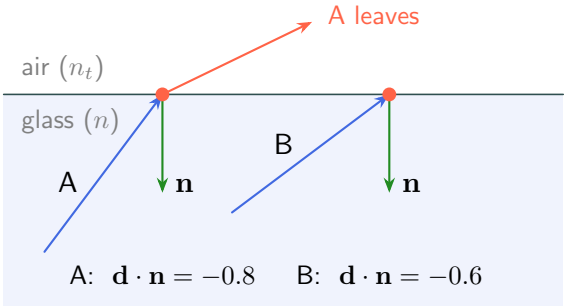
  $$1 - \frac{n^2}{n_t^2}\left(1 - (\mathbf{d} \cdot \mathbf{n})^2\right)$$

  $n = 1.5, \;\; n_t = 1: \;\; n^2/n_t^2 = 2.25$

  A: $\;1 - 2.25\,(1 - 0.8^2) = 0.19$

Worked: can these two rays leave?



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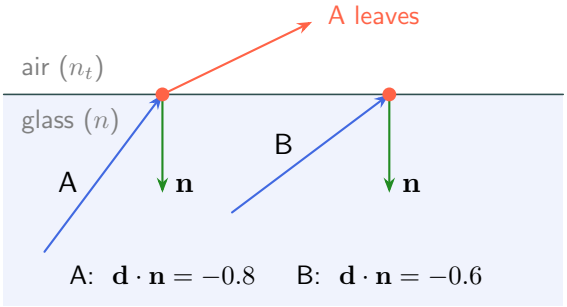
  $$1 - \frac{n^2}{n_t^2}\left(1 - (\mathbf{d}\cdot\mathbf{n})^2\right)$$

  $n = 1.5,\ \ n_t = 1:\ \ n^2/n_t^2 = 2.25$

  A: $1 - 2.25\,(1 - 0.8^2) = 0.19$

  positive: a bent ray, A leaves

Worked: can these two rays leave?



- Try first: which ray leaves the glass?

  the number under the formula's root:

  $$1 - \frac{n^2}{n_t^2}\left(1 - (\mathbf{d}\cdot\mathbf{n})^2\right)$$

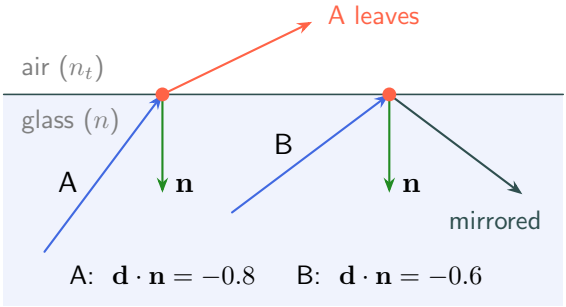
  $n = 1.5,\ \ n_t = 1:\ \ n^2/n_t^2 = 2.25$

  A: $1 - 2.25\,(1 - 0.8^2) = 0.19$

  positive: a bent ray, A leaves

  B: $1 - 2.25\,(1 - 0.6^2) = -0.44$

Worked: can these two rays leave?



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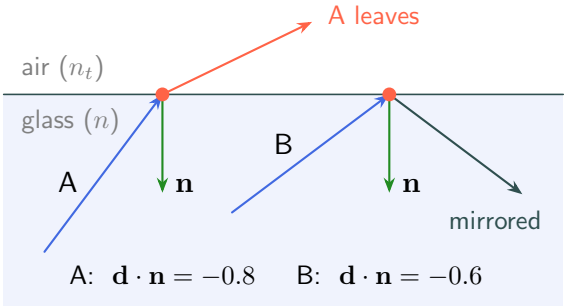
  positive: a bent ray, A leaves

  B: $1 - 2.25\,(1 - 0.6^2) = -0.44$

  negative: no bent ray, mirrored

Worked: can these two rays leave?



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  A: $1 - 2.25\,(1 - 0.8^2) = 0.19$

  positive: a bent ray, A leaves

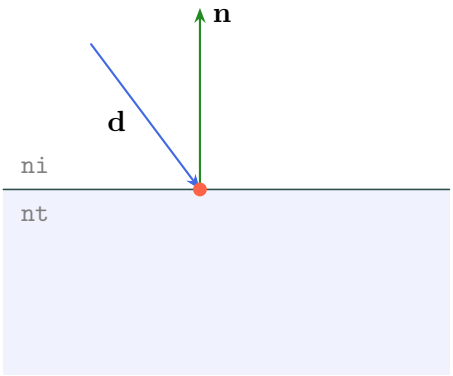
  B: $1 - 2.25\,(1 - 0.6^2) = -0.44$

  negative: no bent ray, mirrored

- Check the sign before the root

Refraction, in Python

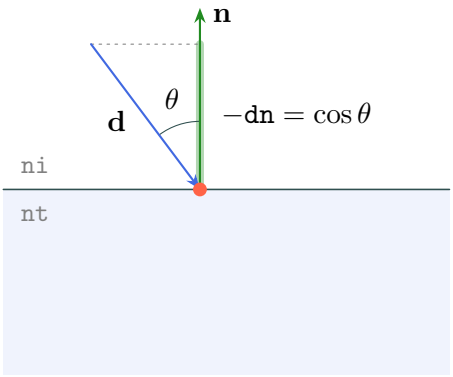
```
def refract(d, n, ni, nt):
```



- `d` and `n` have length 1; `n` faces the ray.
- `ni` is the ray's side, `nt` the far side.

Refraction, in Python

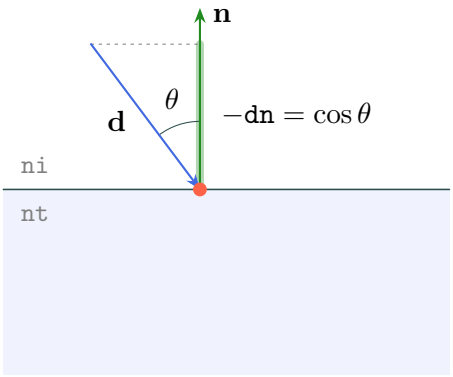
```
def refract(d, n, ni, nt):
    dn = d.dot(n)
```



- `d` and `n` have length 1; `n` faces the ray.
- `ni` is the ray's side, `nt` the far side.

Refraction, in Python

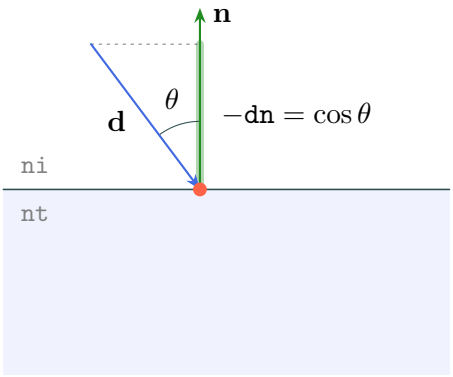
```
def refract(d, n, ni, nt):
    dn = d.dot(n)
    under = 1 - ni**2 * (1 - dn**2) / nt**2
```



- `d` and `n` have length 1; `n` faces the ray.
- `ni` is the ray's side, `nt` the far side.
- `under` is the number under the root.

Refraction, in Python

```
def refract(d, n, ni, nt):
    dn = d.dot(n)
    under = 1 - ni**2 * (1 - dn**2) / nt**2
    if under < 0:
        return None
```



- `d` and `n` have length 1; `n` faces the ray.
- `ni` is the ray's side, `nt` the far side.
- `under` is the number under the root.
- `None` tells the caller to mirror the ray.

Refraction, in Python

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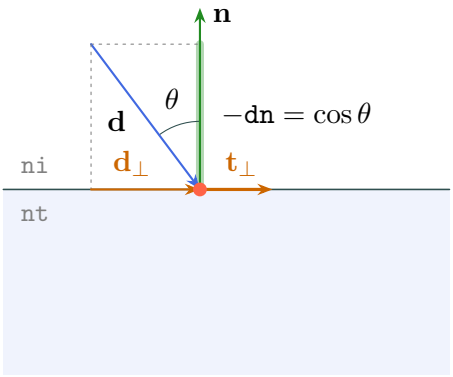
def refract(d, n, ni, nt):
    dn = d.dot(n)
    under = 1 - ni**2 * (1 - dn**2) / nt**2
    if under < 0:
        return None
    t_perp = (ni / nt) * (d - dn * n)
```



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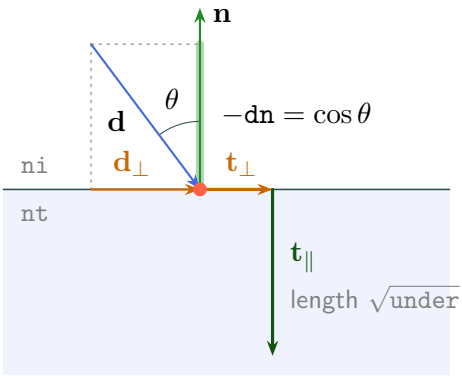

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    t_parallel = -math.sqrt(under) * n
    return t_perp + t_parallel
```

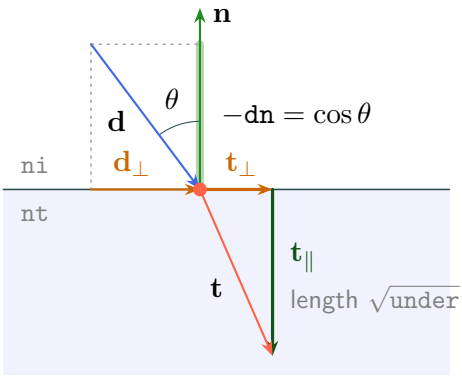

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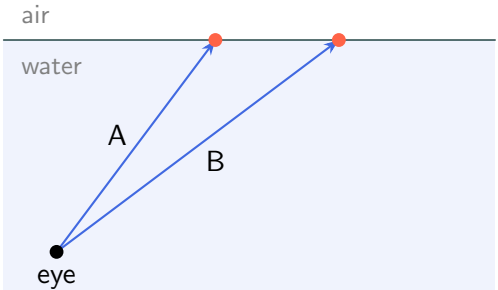
Quick check 2: looking up from under water



- From below: part window, part mirror

Water into air: $n = 1.33,\ n_t = 1$

A: $\sin\theta = 0.6$ B: $\sin\theta = 0.8$

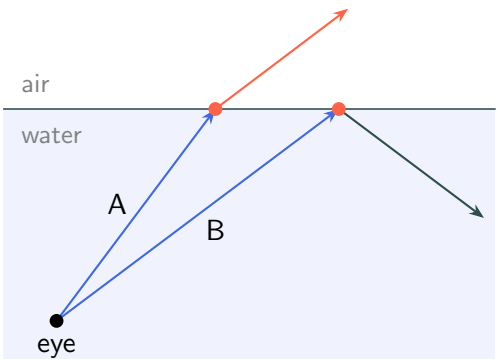
1. Which ray gets out into the air?

   _____

2. What does the eye see along the other?

   _____

Quick check 2: looking up from under water



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Water into air: $n = 1.33,\ n_t = 1$

A: $\sin\theta = 0.6$ B: $\sin\theta = 0.8$

1. Which ray gets out into the air?

   A: $1.33 \times 0.6 = 0.80.$ B: $1.06 > 1$

2. What does the eye see along the other?

   the pool, mirrored in the surface

Recap: the test under the root

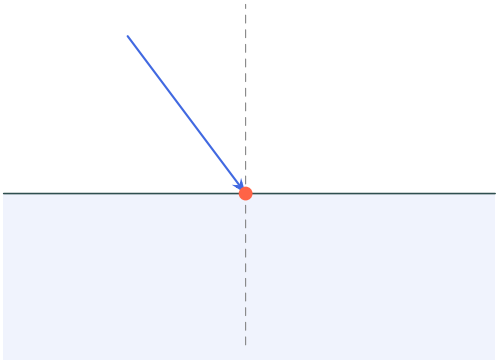
$$1 - \frac{n^2\left(1 - (\mathbf{d}\cdot\mathbf{n})^2\right)}{n_t^2} < 0 \iff \text{no bent ray: the ray is mirrored}$$

only possible on the way out, when $n > n_t$

Two rays at every hit

How much is mirrored, and how the rays multiply

Glass also mirrors: more at a grazing angle



$\theta = 36.87°$, our hit

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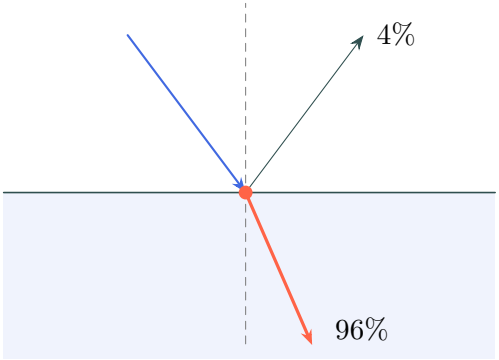

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- Steep: almost all goes through

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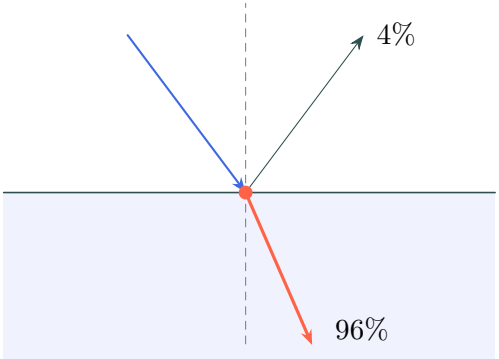

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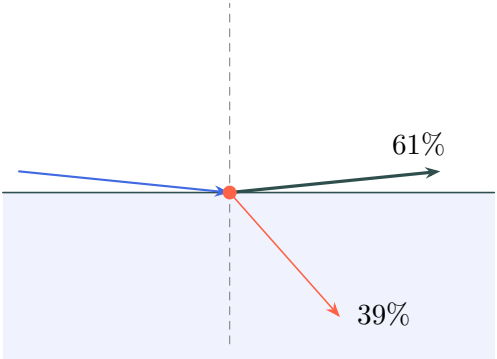
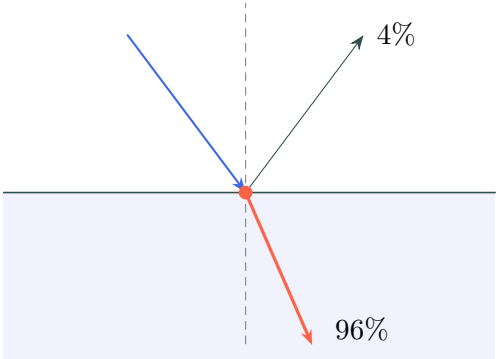


$\theta = 84°$, a grazing ray

- Steep: almost all goes through
- Grazing: most is mirrored

Glass also mirrors: more at a grazing angle



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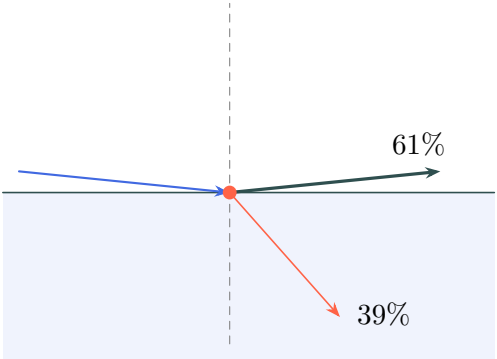
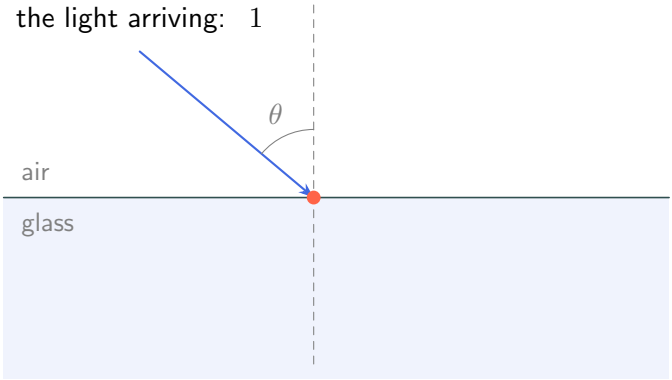


$\theta = 84°$, a grazing ray

- Steep: almost all goes through
- Grazing: most is mirrored
- Like a lake

R: the fraction of the light that is mirrored



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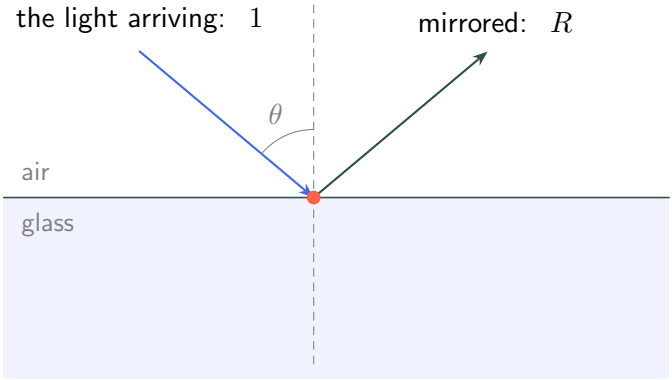
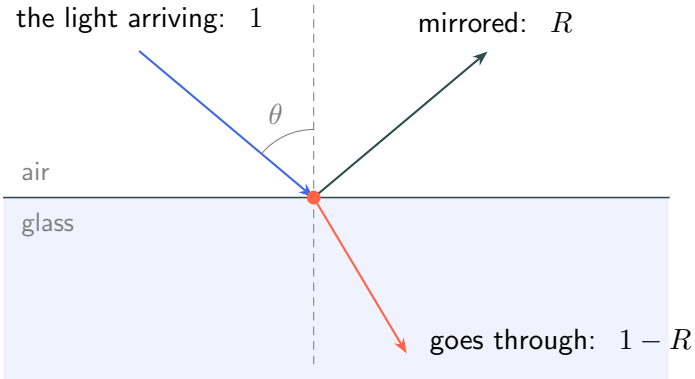


- R: a number from 0 to 1

R: the fraction of the light that is mirrored



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- The rest, $1 - R$, goes through: nothing is lost

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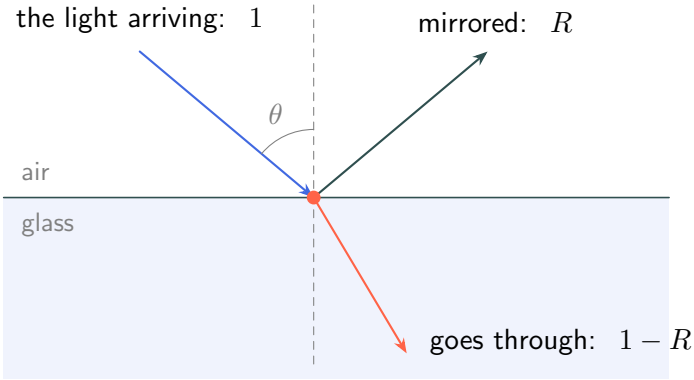


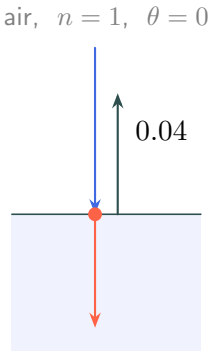
- R: a number from 0 to 1
- The rest, $1 - R$, goes through: nothing is lost
- R changes with θ: 0.04 at $37°$, 0.61 at $84°$
- Wanted: a formula for R

Seen in the world: clear when steep, a mirror at grazing





Head-on, the mirrored fraction is R_0

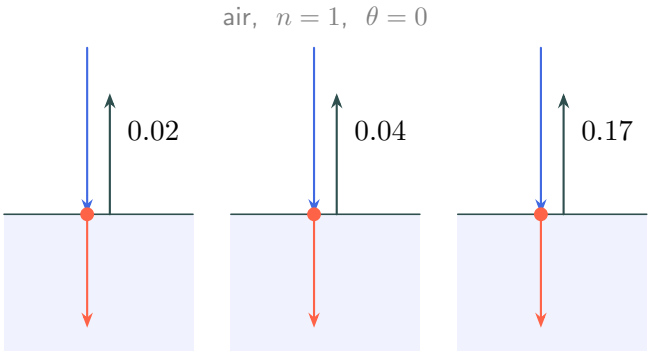


$$R_0 = 0.04$$

$$R_0 = \left(\frac{n_t - n}{n_t + n}\right)^2$$

- Head-on: the ray arrives along the normal
- Only the two indices count

Head-on, the mirrored fraction is R_0



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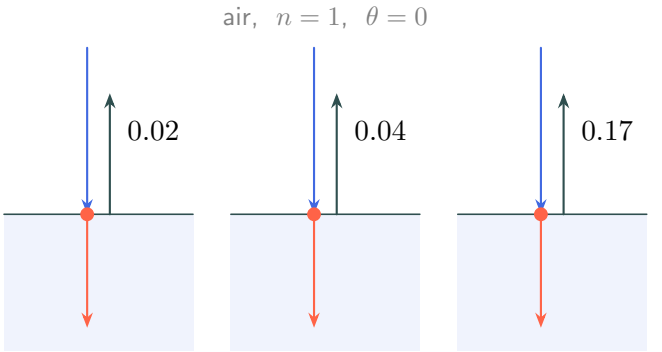

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- Head-on: the ray arrives along the normal
- Only the two indices count
- Indices further apart: more is mirrored
- Equal indices: $R_0 = 0$, nothing is mirrored

For reference: the exact law, Fresnel's equations

$$R_s = \left(\frac{n \cos\theta - n_t \cos\phi}{n \cos\theta + n_t \cos\phi} \right)^2 \qquad R_p = \left(\frac{n \cos\phi - n_t \cos\theta}{n \cos\phi + n_t \cos\theta} \right)^2$$

$$\cos\phi = \sqrt{1 - \frac{n^2 \left(1 - \cos^2\theta\right)}{n_t^2}}$$

$$R = \frac{R_s + R_p}{2}$$

R_s, R_p: the two polarisations of light for reference only

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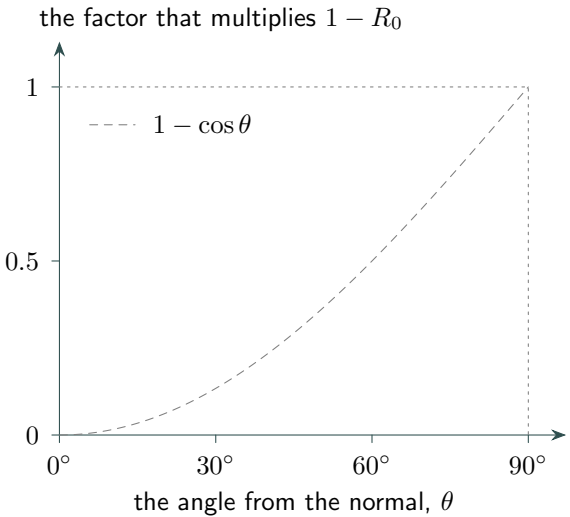
R_s, R_p: the two polarisations of light for reference only

- Schlick, in place of all this: $R = R_0 + (1 - R_0)\,(1 - \cos\theta)^5$

Schlick's number: $(1 - \cos\theta)^5$

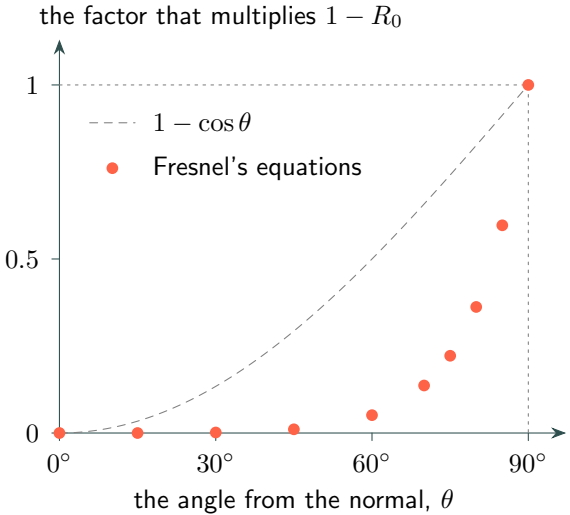


$$R = R_0 + (1 - R_0)\ \underbrace{(1 - \cos\theta)^5}_{\text{from 0 to 1}}$$

- $1 - \cos\theta$: 0 head-on, 1 at grazing

Fresnel's equations: the exact law, but long

Schlick's number: $(1 - \cos\theta)^5$



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- But Fresnel's equations rise much later

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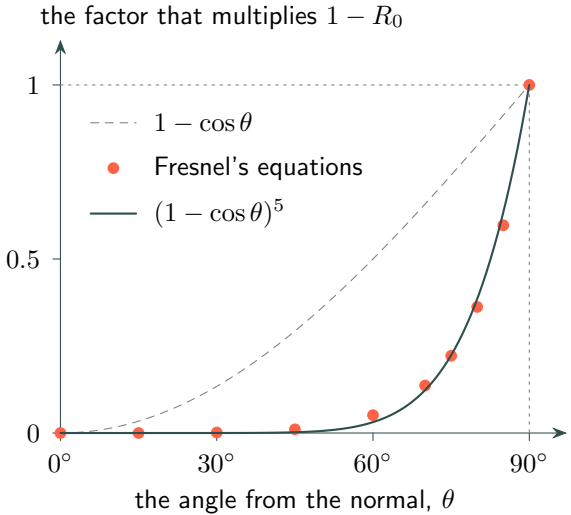

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- To the power 5: still 0 and 1 at the ends, small in between

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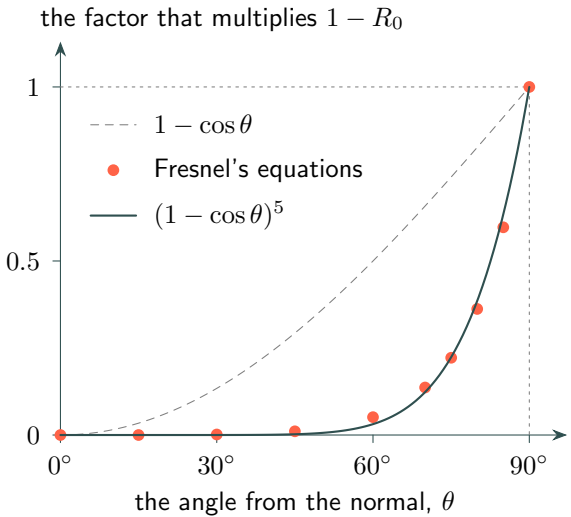

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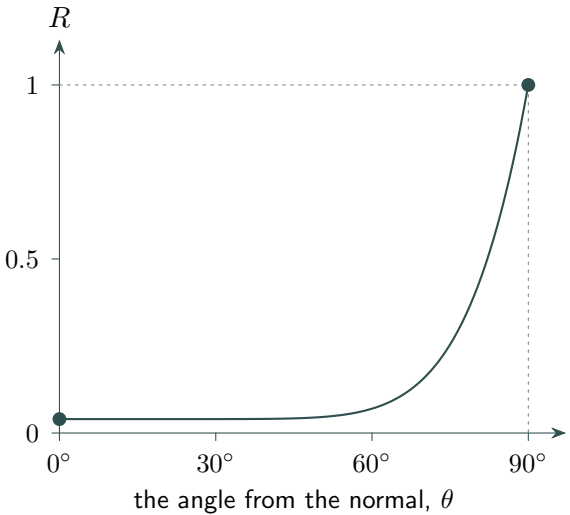
- But Fresnel's equations rise much later
- To the power 5: still 0 and 1 at the ends, small in between
- The 5 is a fit to Fresnel's equations, not derived

Fresnel's equations: the exact law, but long

The fifth power keeps R low, then lifts it fast



$$R = R_0 + (1 - R_0)\,(1 - \cos\theta)^5, \quad R_0 = 0.04$$

θ	$\cos\theta$	$(1-\cos\theta)^5$	R
$0°$	1	0	0.04
$90°$	0	1	1

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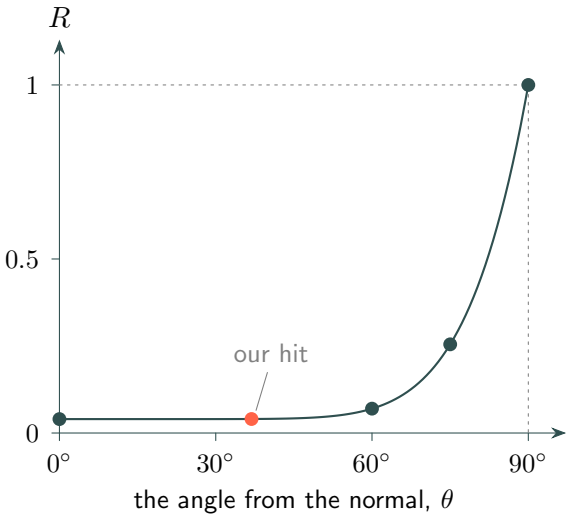

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37°	0.8	0.0003	0.04
60°	0.5	0.03	0.07
75°	0.26	0.22	0.25
90°	0	1	1

- Up to 60°: R hardly moves

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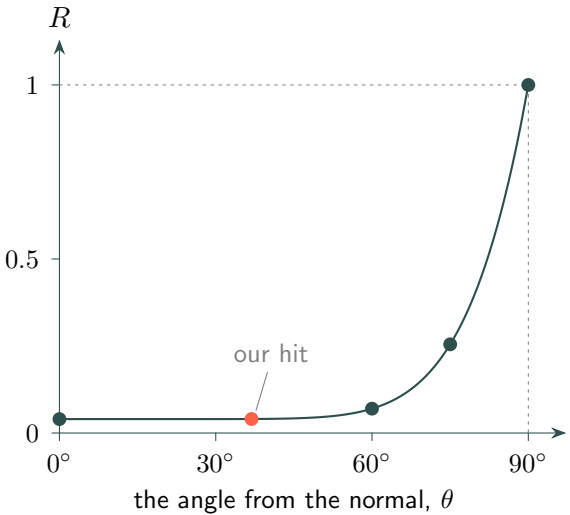

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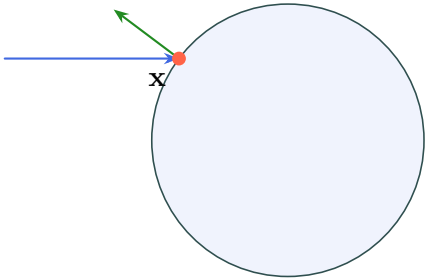
90°	0	1	1

- Up to 60°: R hardly moves
- Past 75°: glass turns into a mirror

Worked: the mirrored fraction at our hit

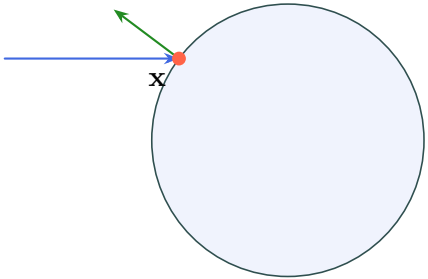


side view

- Try first: R_0, then R at $\mathbf{x}$

$$n = 1, \quad n_t = 1.5, \quad \cos\theta = 0.8$$

Worked: the mirrored fraction at our hit



side view

- Try first: R_0, then R at $\mathbf{x}$

$$n = 1, \quad n_t = 1.5, \quad \cos\theta = 0.8$$

$$R_0 = \left(\frac{1.5-1}{1.5+1}\right)^2 = 0.2^2 = 0.04$$

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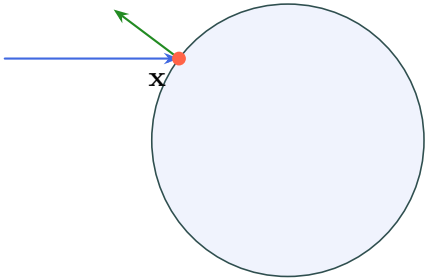

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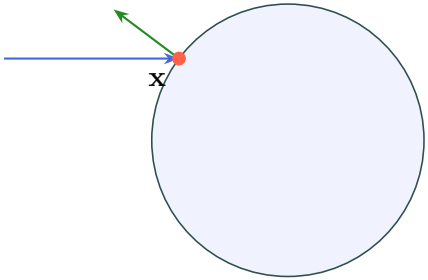
$$(1 - 0.8)^5 = 0.2^5 = 0.00032$$

Worked: the mirrored fraction at our hit



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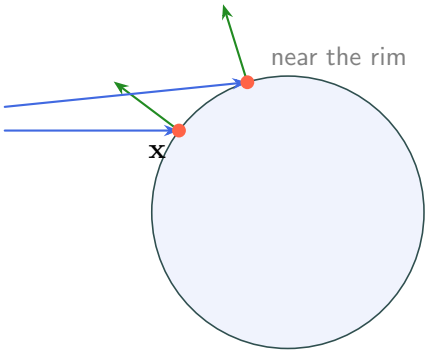
$$R = 0.04 + 0.96 \times 0.00032 = 0.0403$$

Worked: the mirrored fraction at our hit



side view

- Try first: R_0, then R at $\mathbf{x}$

$n = 1, \;\; n_t = 1.5, \;\; \cos\theta = 0.8$

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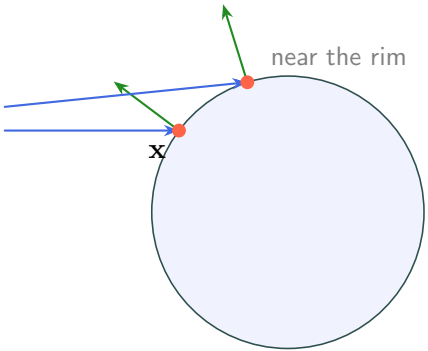
$(1 - 0.8)^5 = 0.2^5 = 0.00032$

$R = 0.04 + 0.96 \times 0.00032 = 0.0403$

near the rim, $\cos\theta = 0.2$: $\;\; 0.8^5 = 0.328$

Worked: the mirrored fraction at our hit



side view

- Try first: R_0, then R at $\mathbf{x}$

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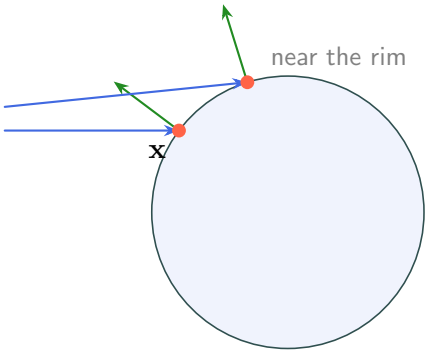
$R = 0.04 + 0.96 \times 0.00032 = 0.0403$

near the rim, $\cos\theta = 0.2$: $\;\; 0.8^5 = 0.328$

$R = 0.04 + 0.96 \times 0.328 = 0.355$

Worked: the mirrored fraction at our hit



side view

- Try first: R_0, then R at $\mathbf{x}$

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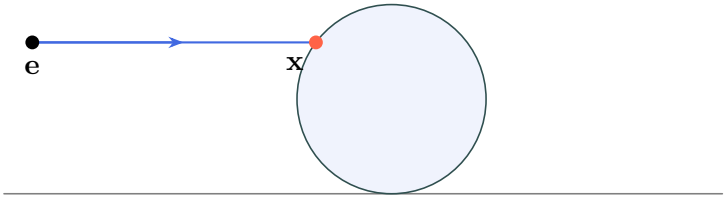
  $R = 0.04 + 0.96 \times 0.00032 = 0.0403$

  near the rim, $\cos\theta = 0.2$: $\;\; 0.8^5 = 0.328$

  $R = 0.04 + 0.96 \times 0.328 = 0.355$

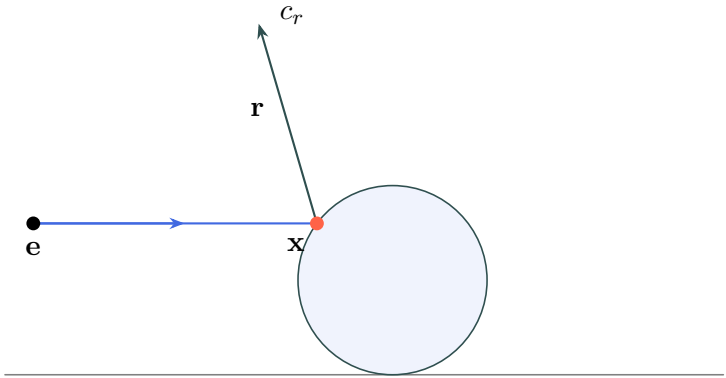
- Middle: 4% mirrored. Rim: 35%

One ray in, two rays out



side view

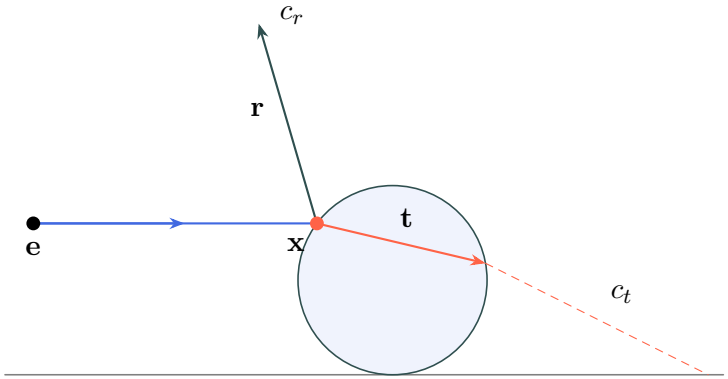
One ray in, two rays out



side view

- Mirrored ray: lecture 17's $\mathbf{r}$

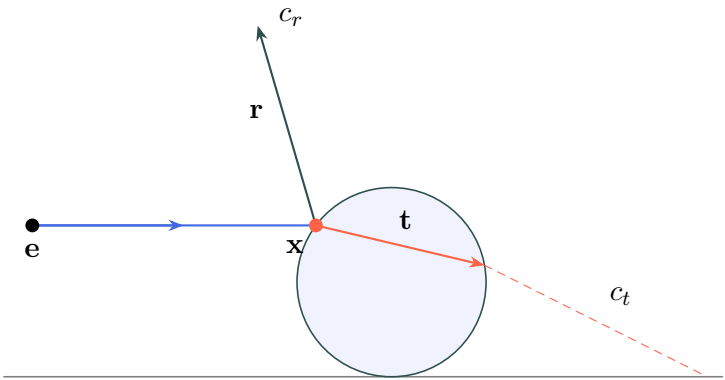
One ray in, two rays out



side view

- Mirrored ray: lecture 17's $\mathbf{r}$
- Bent ray: today's $\mathbf{t}$

One ray in, two rays out



- Mirrored ray: lecture 17's $\mathbf{r}$
- Bent ray: today's $\mathbf{t}$
- Blend their colours with R

$$c = R\,c_r + (1 - R)\,c_t$$

A glass hit adds up no lamps. Its whole colour comes from these two rays.

side view

Worked: the middle pixel, through the glass



- Try first: the pixel's colour

$$R = 0.04, \quad c_r = (0.4,\ 0.6,\ 1.0), \quad c_t = (0.326,\ 0.326,\ 0.327)$$

Worked: the middle pixel, through the glass



- Try first: the pixel's colour

$$R = 0.04, \quad c_r = (0.4,\ 0.6,\ 1.0), \quad c_t = (0.326,\ 0.326,\ 0.327)$$

$$0.04\, c_r = (0.016,\ 0.024,\ 0.040)$$

Worked: the middle pixel, through the glass



- Try first: the pixel's colour

$$R = 0.04, \quad c_r = (0.4,\ 0.6,\ 1.0), \quad c_t = (0.326,\ 0.326,\ 0.327)$$

$$0.04\, c_r = (0.016,\ 0.024,\ 0.040) \qquad 0.96\, c_t = (0.313,\ 0.313,\ 0.314)$$

Worked: the middle pixel, through the glass



- Try first: the pixel's colour

$$R = 0.04, \quad c_r = (0.4,\ 0.6,\ 1.0), \quad c_t = (0.326,\ 0.326,\ 0.327)$$

$$0.04\, c_r = (0.016,\ 0.024,\ 0.040) \qquad 0.96\, c_t = (0.313,\ 0.313,\ 0.314)$$

$$c = (0.329,\ 0.337,\ 0.354)$$

Worked: the middle pixel, through the glass



- Try first: the pixel's colour

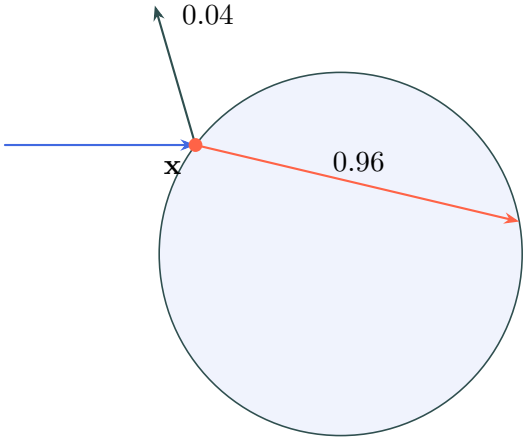
$R = 0.04, \quad c_r = (0.4,\ 0.6,\ 1.0), \quad c_t = (0.326,\ 0.326,\ 0.327)$

$0.04\, c_r = (0.016,\ 0.024,\ 0.040) \qquad 0.96\, c_t = (0.313,\ 0.313,\ 0.314)$

$c = (0.329,\ 0.337,\ 0.354)$

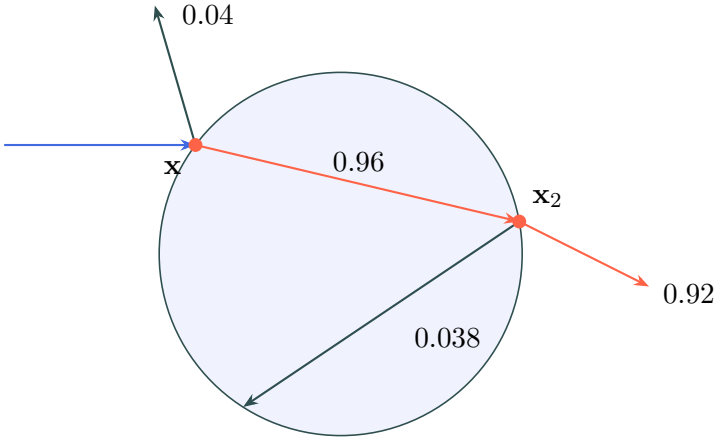
- Almost all of it: the bent ray

Inside the ball, the ray splits again



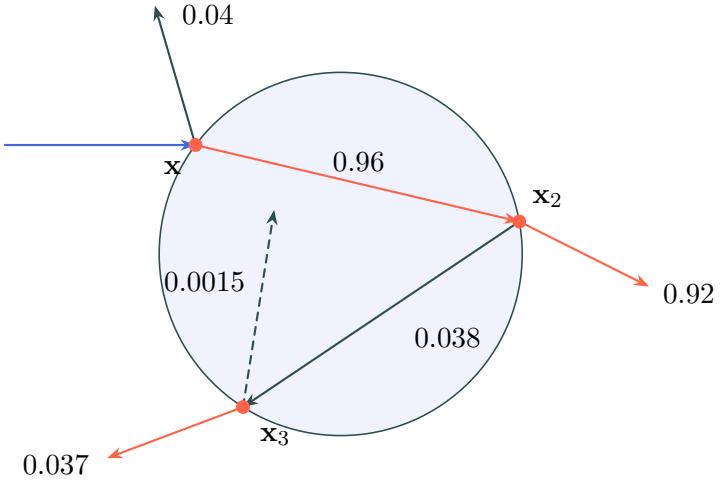
- At $\mathbf{x}$: 0.04 mirrored, 0.96 in

Inside the ball, the ray splits again



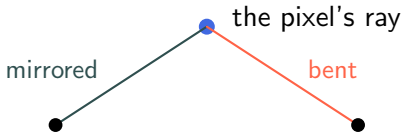
- At $\mathbf{x}$: 0.04 mirrored, 0.96 in
- At $\mathbf{x}_2$: one out, one stays
- Fractions multiply: $0.96 \times 0.96 = 0.92$

Inside the ball, the ray splits again



- At $\mathbf{x}$: 0.04 mirrored, 0.96 in
- At $\mathbf{x}_2$: one out, one stays
- Fractions multiply: $0.96 \times 0.96 = 0.92$
- Two bounces inside: almost nothing

A tree of rays needs a depth limit



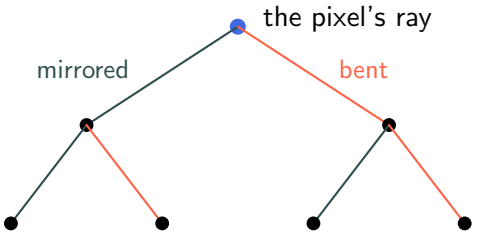
1 ray

2 rays

- Glass: two new rays per hit

the worst case: every ray meets glass again

A tree of rays needs a depth limit



1 ray

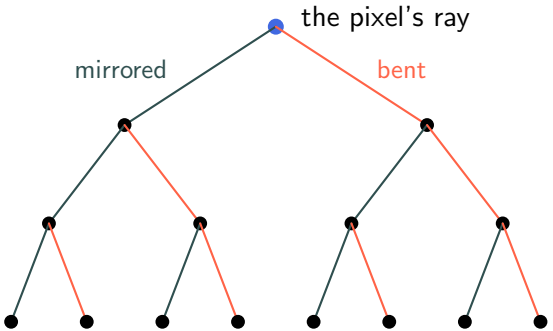
2 rays

4 rays

- Glass: two new rays per hit
- Each level doubles: 1, 2, 4, 8

the worst case: every ray meets glass again

A tree of rays needs a depth limit



1 ray

2 rays

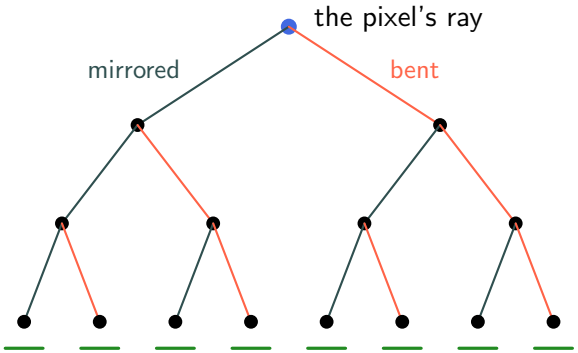
4 rays

8 rays

- Glass: two new rays per hit
- Each level doubles: 1, 2, 4, 8

the worst case: every ray meets glass again

A tree of rays needs a depth limit



1 ray

2 rays

4 rays

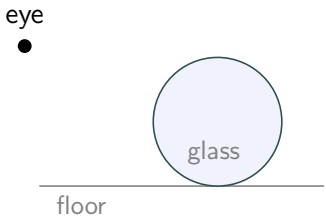
8 rays

- Glass: two new rays per hit
- Each level doubles: 1, 2, 4, 8
- Lecture 17's depth counter ends it
- Depth 0: no new ray, no light

the worst case: every ray meets glass again

trace gets one more test: glass

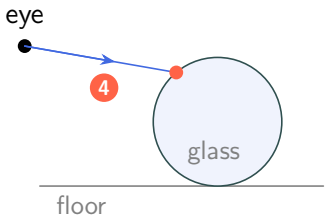
```
def trace(ray, depth):
    rec = scene.hit(ray)
    if rec.t == math.inf:
        return the sky's colour          1
    material = rec.surface.material
    if isinstance(material, (Lambertian, BlinnPhong)):
        return the lamps' shares, added up   2
    elif isinstance(material, Reflective):
        return the mirror's colour       3
```



Grey: lecture 17's `trace`, unchanged.

trace gets one more test: glass

```
def trace(ray, depth):
    rec = scene.hit(ray)
    if rec.t == math.inf:
        return the sky's colour          1
    material = rec.surface.material
    if isinstance(material, (Lambertian, BlinnPhong)):
        return the lamps' shares, added up   2
    elif isinstance(material, Reflective):
        return the mirror's colour       3
    elif isinstance(material, Glass):
        return shade_glass(ray, rec, depth)  4
```

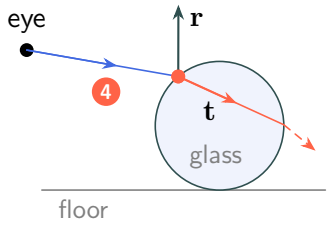


4 The surface is `Glass`. Return the colour from `shade_glass` (next slide).

Grey: lecture 17's `trace`, unchanged.

trace gets one more test: glass

```
def trace(ray, depth):
    rec = scene.hit(ray)
    if rec.t == math.inf:
        return the sky's colour          1
    material = rec.surface.material
    if isinstance(material, (Lambertian, BlinnPhong)):
        return the lamps' shares, added up   2
    elif isinstance(material, Reflective):
        return the mirror's colour       3
    elif isinstance(material, Glass):
        return shade_glass(ray, rec, depth)  4
```



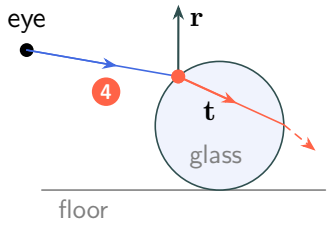
4 The surface is `Glass`. Return the colour from `shade_glass` (next slide).

- A glass hit adds up no lamps. Its colour comes from two more rays.

Grey: lecture 17's `trace`, unchanged.

trace gets one more test: glass

```
def trace(ray, depth):
    rec = scene.hit(ray)
    if rec.t == math.inf:
        return the sky's colour                    1
    material = rec.surface.material
    if isinstance(material, (Lambertian, BlinnPhong)):
        return the lamps' shares, added up         2
    elif isinstance(material, Reflective):
        return the mirror's colour                 3
    elif isinstance(material, Glass):
        return shade_glass(ray, rec, depth)        4
```



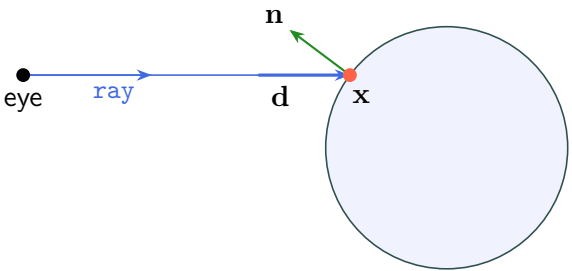
4 The surface is `Glass`. Return the colour from `shade_glass` (next slide).

- A glass hit adds up no lamps. Its colour comes from two more rays.
- `Glass` is one more kind of material. It stores one number, `refractive_index`.

Grey: lecture 17's `trace`, unchanged. The camera now calls `trace(ray, 6)`.

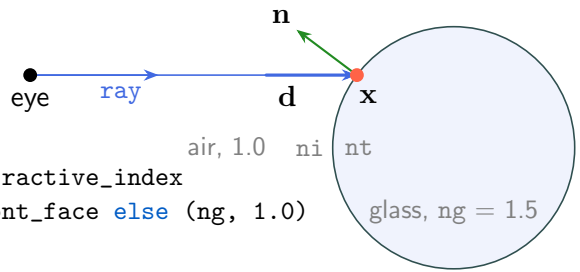
Glass, in Python

```
def shade_glass(ray, rec, depth):
    if depth == 0:
        return Vec3(0,0,0)
    x = ray.evaluate(rec.t)
    d = ray.d / ray.d.length()
    n = rec.normal
```



Glass, in Python

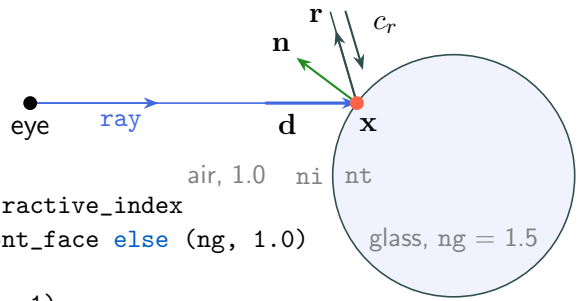
```
def shade_glass(ray, rec, depth):
    if depth == 0:
        return Vec3(0,0,0)
    x = ray.evaluate(rec.t)
    d = ray.d / ray.d.length()
    n = rec.normal
    ng = rec.surface.material.refractive_index
    ni, nt = (1.0, ng) if rec.front_face else (ng, 1.0)
```



- The other side is taken to be air. A hit from inside swaps `ni` and `nt`.

Glass, in Python

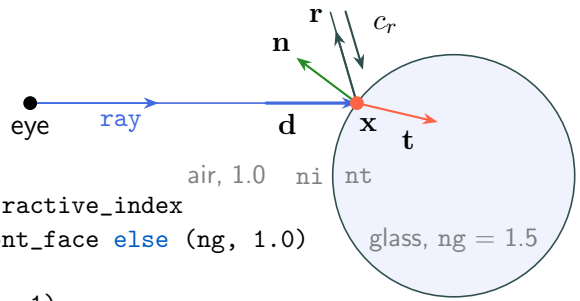
```
def shade_glass(ray, rec, depth):
    if depth == 0:
        return Vec3(0,0,0)
    x = ray.evaluate(rec.t)
    d = ray.d / ray.d.length()
    n = rec.normal
    ng = rec.surface.material.refractive_index
    ni, nt = (1.0, ng) if rec.front_face else (ng, 1.0)
    r = d - 2 * d.dot(n) * n
    c_r = trace(Ray(x, r), depth - 1)
```



- The other side is taken to be air. A hit from inside swaps `ni` and `nt`.

Glass, in Python

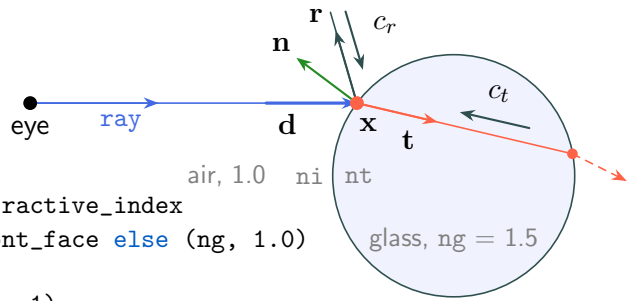
```
def shade_glass(ray, rec, depth):
    if depth == 0:
        return Vec3(0,0,0)
    x = ray.evaluate(rec.t)
    d = ray.d / ray.d.length()
    n = rec.normal
    ng = rec.surface.material.refractive_index
    ni, nt = (1.0, ng) if rec.front_face else (ng, 1.0)
    r = d - 2 * d.dot(n) * n
    c_r = trace(Ray(x, r), depth - 1)
    t = refract(d, n, ni, nt)
    if t is None:  return c_r
```



- The other side is taken to be air. A hit from inside swaps `ni` and `nt`.
- If `t` is `None`, all the light is mirrored.

Glass, in Python

```
def shade_glass(ray, rec, depth):
    if depth == 0:
        return Vec3(0,0,0)
    x = ray.evaluate(rec.t)
    d = ray.d / ray.d.length()
    n = rec.normal
    ng = rec.surface.material.refractive_index
    ni, nt = (1.0, ng) if rec.front_face else (ng, 1.0)
    r = d - 2 * d.dot(n) * n
    c_r = trace(Ray(x, r), depth - 1)
    t = refract(d, n, ni, nt)
    if t is None:  return c_r
    c_t = trace(Ray(x, t), depth - 1)
```



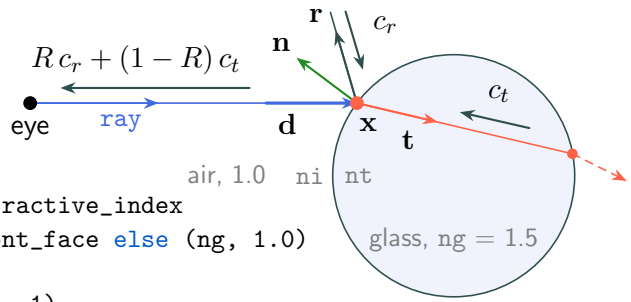
- The other side is taken to be air. A hit from inside swaps `ni` and `nt`.
- If `t` is `None`, all the light is mirrored.

- `trace` is called twice, each time with depth -1.

Glass, in Python

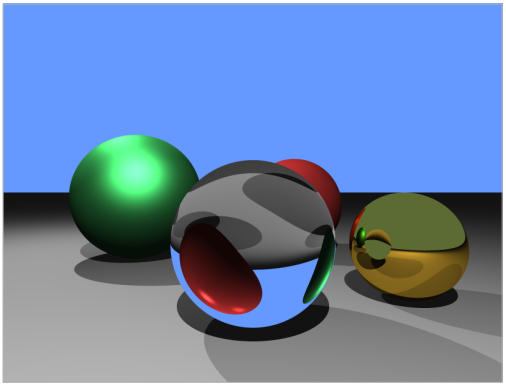
```
def shade_glass(ray, rec, depth):
    if depth == 0:
        return Vec3(0,0,0)
    x = ray.evaluate(rec.t)
    d = ray.d / ray.d.length()
    n = rec.normal
    ng = rec.surface.material.refractive_index
    ni, nt = (1.0, ng) if rec.front_face else (ng, 1.0)
    r = d - 2 * d.dot(n) * n
    c_r = trace(Ray(x, r), depth - 1)
    t = refract(d, n, ni, nt)
    if t is None:  return c_r
    c_t = trace(Ray(x, t), depth - 1)
    R = schlick(-d.dot(n), ni, nt)
    return R * c_r + (1 - R) * c_t
```

- `trace` is called twice, each time with depth -1.



- The other side is taken to be air. A hit from inside swaps `ni` and `nt`.
- If `t` is `None`, all the light is mirrored.

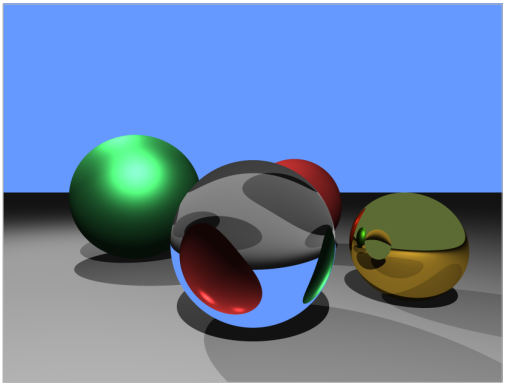
The picture: bent ray only, then both rays



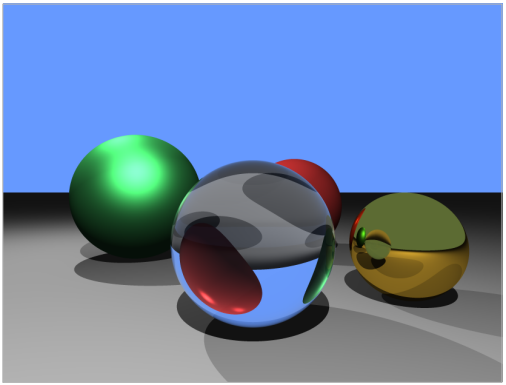
the bent ray only

- The scene, upside down

The picture: bent ray only, then both rays



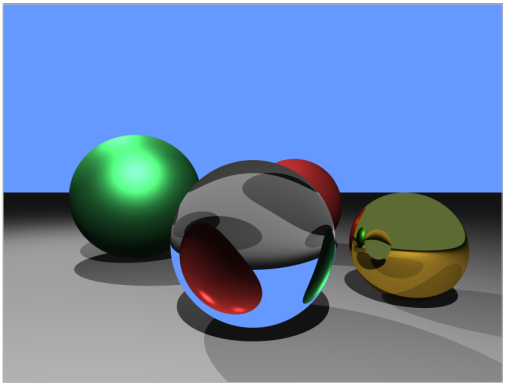
the bent ray only



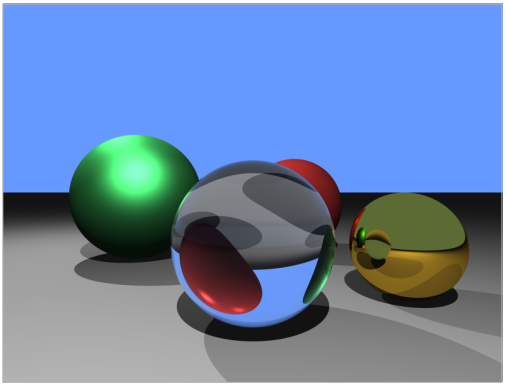
both rays, blended with R

- The scene, upside down
- Right: a pale rim

The picture: bent ray only, then both rays



the bent ray only



both rays, blended with R

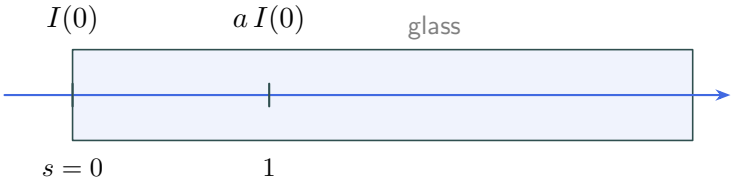
- The scene, upside down
- Right: a pale rim
- The middle: hardly changed

Coloured glass dims light along the path



- I: the light a ray carries

Coloured glass dims light along the path

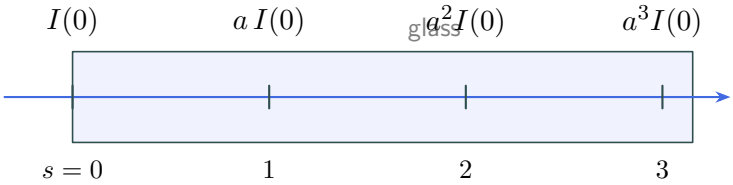


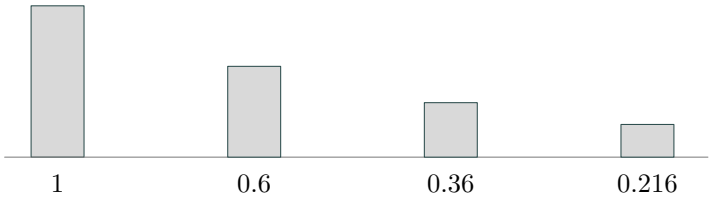


what is left, with $a = 0.6$

- I: the light a ray carries
- a: what one unit leaves

Coloured glass dims light along the path

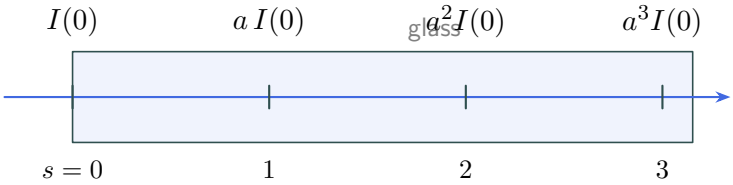


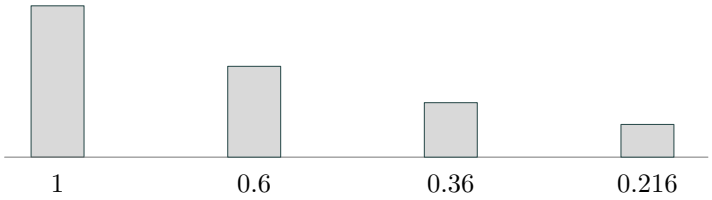


what is left, with $a = 0.6$

- I: the light a ray carries
- a: what one unit leaves
- Every further unit: times a

Coloured glass dims light along the path





what is left, with $a = 0.6$

- I: the light a ray carries
- a: what one unit leaves
- Every further unit: times a

$$I(s) = I(0)\,a^s$$

Beer's law

- One a per channel: a tint

Worked: Beer's law along our ray

- Try first: what is left of each colour

$a = (0.6,\ 0.9,\ 0.6)$ path inside: $s = 1.833$

$\mathbf{x}$ to $\mathbf{x}_2$: $2\cos\phi = 2 \times 0.9165$

Worked: Beer's law along our ray

- Try first: what is left of each colour

$a = (0.6,\ 0.9,\ 0.6)$ path inside: $s = 1.833$

$\mathbf{x}$ to $\mathbf{x}_2$: $2\cos\phi = 2 \times 0.9165$

red, blue: $0.6^{1.833} = 0.392$

Worked: Beer's law along our ray

- Try first: what is left of each colour

$a = (0.6,\ 0.9,\ 0.6)$ path inside: $s = 1.833$

$\mathbf{x}$ to $\mathbf{x}_2$: $2\cos\phi = 2 \times 0.9165$

red, blue: $0.6^{1.833} = 0.392$

green: $0.9^{1.833} = 0.824$

Worked: Beer's law along our ray

- Try first: what is left of each colour

$a = (0.6,\ 0.9,\ 0.6)$ path inside: $s = 1.833$

$\mathbf{x}$ to $\mathbf{x}_2$: $2\cos\phi = 2 \times 0.9165$

red, blue: $0.6^{1.833} = 0.392$

green: $0.9^{1.833} = 0.824$

$a^s = (0.39,\ 0.82,\ 0.39)$



Worked: Beer's law along our ray

- Try first: what is left of each colour

$a = (0.6,\ 0.9,\ 0.6)$ path inside: $s = 1.833$

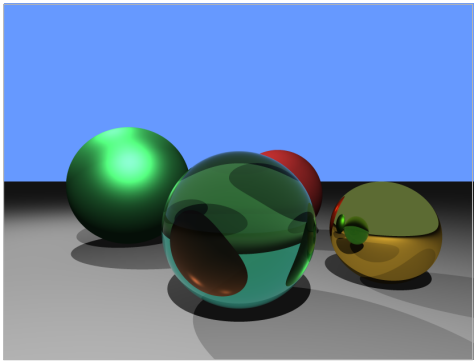
$\mathbf{x}$ to $\mathbf{x}_2$: $2\cos\phi = 2 \times 0.9165$

red, blue: $0.6^{1.833} = 0.392$

green: $0.9^{1.833} = 0.824$

$a^s = (0.39,\ 0.82,\ 0.39)$ 

- Green passes; red and blue fade



the glass ball, with Beer's law

Recap: the colour of a glass hit

$$c \;=\; R\,c_r + (1-R)\,c_t$$

$$R \;=\; R_0 + (1-R_0)\,(1-\cos\theta)^5 \qquad R_0 = \left(\frac{n_t - n}{n_t + n}\right)^2$$

on a glass hit, `trace` asks no lamp and returns this c

two more calls of `trace`; a depth limit ends the tree

Summary

1 Snell's law

$$n \sin\theta = n_t \sin\phi$$

into glass: toward $\mathbf{n}$

2 The bent direction

$$\mathbf{t} = \frac{n\left(\mathbf{d} - \mathbf{n}\left(\mathbf{d}\cdot\mathbf{n}\right)\right)}{n_t} - \mathbf{n}\sqrt{1 - \frac{n^2\left(1 - (\mathbf{d}\cdot\mathbf{n})^2\right)}{n_t^2}}$$

$\mathbf{t}_\perp$ along the surface, plus $\mathbf{t}_\parallel$ along the normal

3 Total internal reflection

under the root < 0: no bent ray

only on the way out, $n > n_t$

4 Two rays, blended

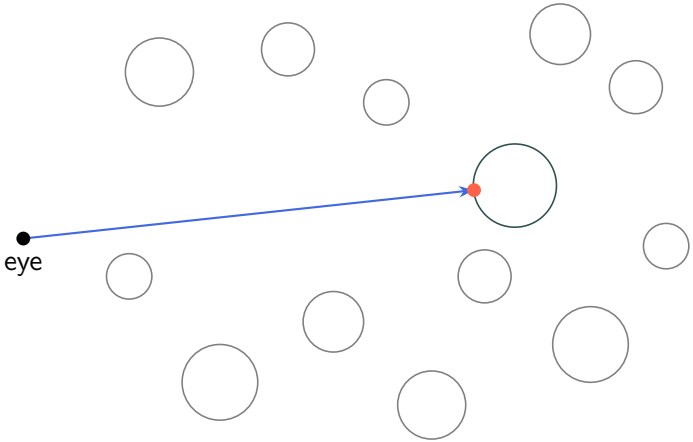
$$c = R\,c_r + (1 - R)\,c_t$$

$$R = R_0 + (1 - R_0)\,(1 - \cos\theta)^5$$

`trace` gets one more `elif`, for glass

a depth limit ends the tree

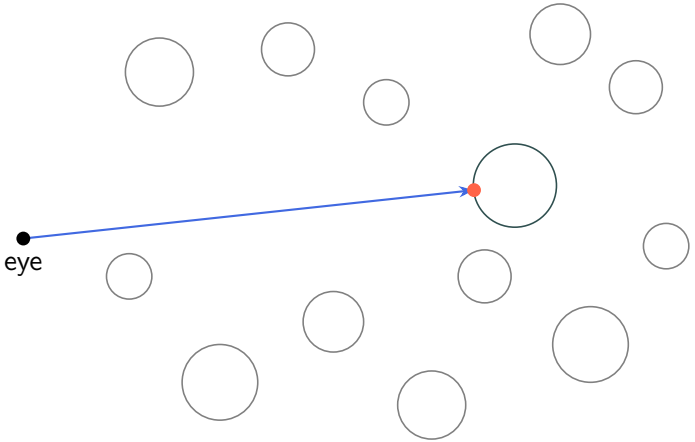
Next lecture: why ray tracing is slow



a sketch: one ray, many balls

- One ray: a test for every ball

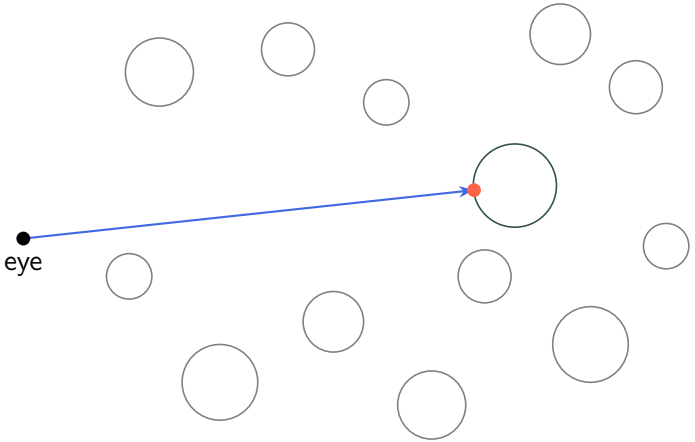
Next lecture: why ray tracing is slow



a sketch: one ray, many balls

- One ray: a test for every ball
- Mirrors and glass: many more rays

Next lecture: why ray tracing is slow



a sketch: one ray, many balls

- One ray: a test for every ball
- Mirrors and glass: many more rays
- Next: where the time goes

Exit Ticket

Today's question *(2 minutes)*

A ray travels from air ($n = 1$) into glass ($n_t = 1.5$). It arrives with $\sin\theta = 0.9$.

(a) Find $\sin\phi$ inside the glass.

(b) Another ray is inside the glass and reaches the surface with $\sin\theta = 0.8$. Does it get out into the air? Show the number you tested.

CS 453 — Exit Ticket #18

Name: _____

Roll #: _____ Date: _____

Answer:

Hand it to me on your way out.
No name, no attendance.