Lecture 14: Ray-Sphere Intersection

Substitute, Solve a Quadratic, Keep the First Hit

CS 453 — Computer Graphics

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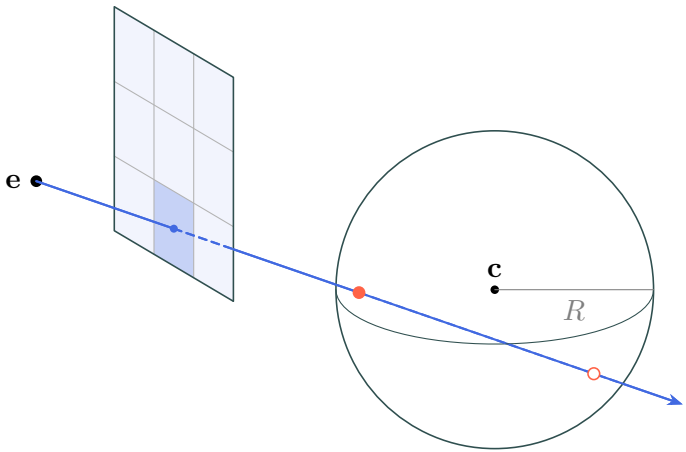
Forman Christian University

Today: where does a ray meet a sphere?

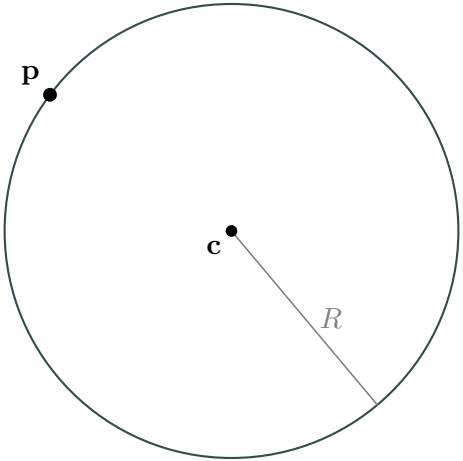


- Lecture 13: one ray per pixel
- Today: a sphere in its way
- Where does the ray go in?

The ray meets the sphere

Put the ray into the sphere's equation

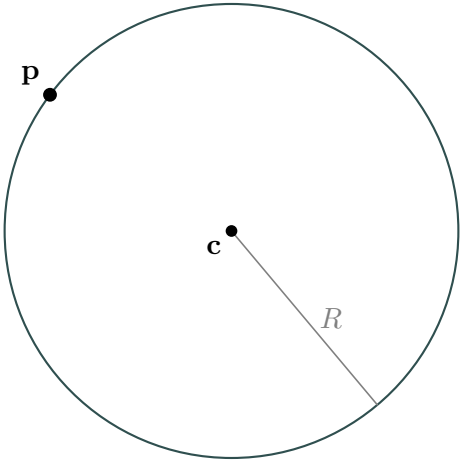
The sphere's equation, as one dot product



seen from the side

$x^2 + y^2 + z^2 = R^2$ centre at the origin

The sphere's equation, as one dot product



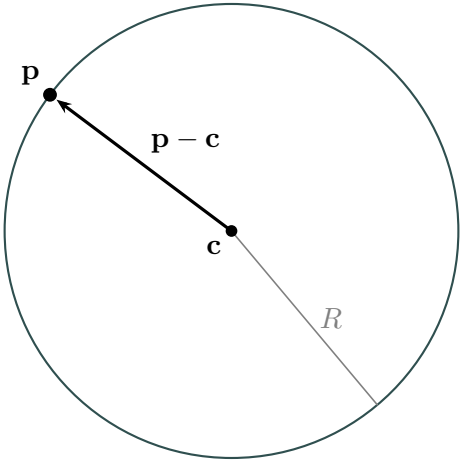
seen from the side

$x^2 + y^2 + z^2 = R^2$ centre at the origin

$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = R^2$

centre $\mathbf{c} = (x_c, y_c, z_c)$: subtract it

The sphere's equation, as one dot product



seen from the side

$x^2 + y^2 + z^2 = R^2$ centre at the origin

$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = R^2$

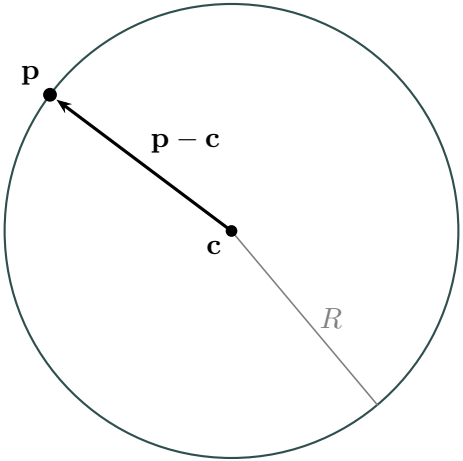
centre $\mathbf{c} = (x_c, y_c, z_c)$: subtract it

- An arrow dotted with itself: squared length

$(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) = R^2$

Ray Tracing in One Weekend: $\mathbf{c} - \mathbf{p}$, same number

The sphere's equation, as one dot product



seen from the side

$x^2 + y^2 + z^2 = R^2$ centre at the origin

$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = R^2$

centre $\mathbf{c} = (x_c, y_c, z_c)$: subtract it

- An arrow dotted with itself: squared length

$(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) = R^2$

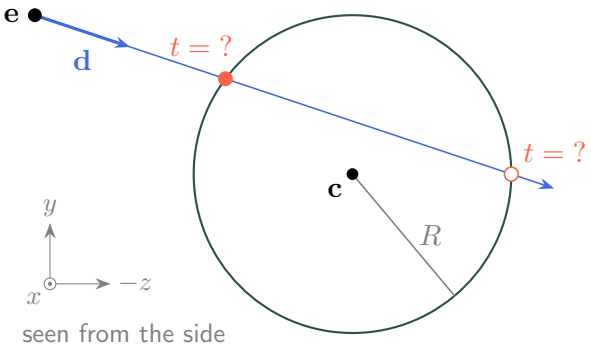
Ray Tracing in One Weekend: $\mathbf{c} - \mathbf{p}$, same number

$\mathbf{p} - \mathbf{c} = (0, 3, -6) - (0, 0, -10) = (0, 3, 4)$

$(0, 3, 4) \cdot (0, 3, 4) = 0 + 9 + 16 = 25 = R^2$

- Any $\mathbf{p}$ that fits is on the sphere

Put the ray into the sphere's equation

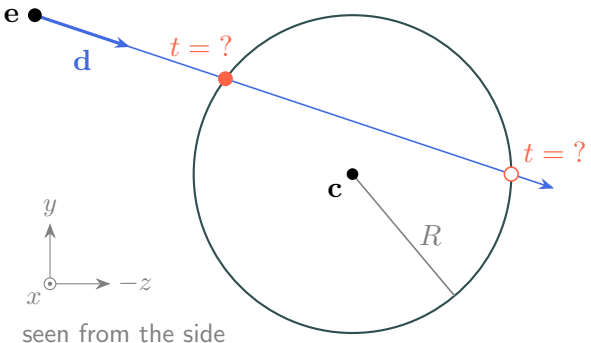


seen from the side

$$(\mathbf{p}(t) - \mathbf{c}) \cdot (\mathbf{p}(t) - \mathbf{c}) = R^2$$

- A hit: some t puts $\mathbf{p}(t)$ on the sphere

Put the ray into the sphere's equation



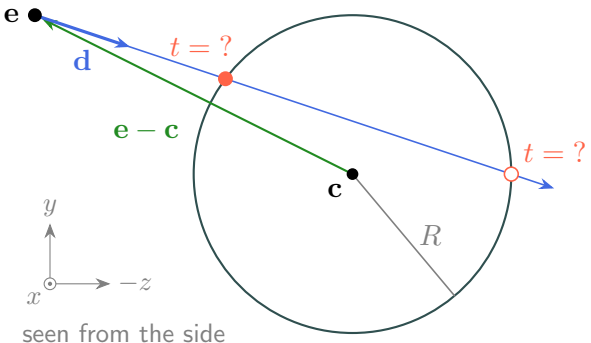
seen from the side

- A hit: some t puts $\mathbf{p}(t)$ on the sphere
- Put in $\mathbf{p}(t) = \mathbf{e} + t\,\mathbf{d}$

$$(\mathbf{p}(t) - \mathbf{c}) \cdot (\mathbf{p}(t) - \mathbf{c}) = R^2$$

$$(\mathbf{e} + t\,\mathbf{d} - \mathbf{c}) \cdot (\mathbf{e} + t\,\mathbf{d} - \mathbf{c}) = R^2$$

Put the ray into the sphere's equation



$$(\mathbf{p}(t) - \mathbf{c}) \cdot (\mathbf{p}(t) - \mathbf{c}) = R^2$$

$$(\mathbf{e} + t\,\mathbf{d} - \mathbf{c}) \cdot (\mathbf{e} + t\,\mathbf{d} - \mathbf{c}) = R^2$$

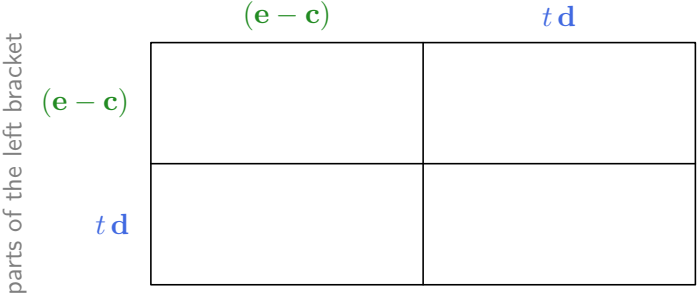
$$\big((\mathbf{e} - \mathbf{c}) + t\,\mathbf{d}\big) \cdot \big((\mathbf{e} - \mathbf{c}) + t\,\mathbf{d}\big) = R^2$$

- A hit: some t puts $\mathbf{p}(t)$ on the sphere
- Put in $\mathbf{p}(t) = \mathbf{e} + t\,\mathbf{d}$
- Group the parts with and without t

Multiply out: four dot products

$$\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)\cdot\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)=R^2$$

parts of the right bracket



- Each part meets each part

Multiply out: four dot products

$$\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)\cdot\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)=R^2$$

parts of the right bracket

parts of the left bracket

	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$	
$t\,\mathbf{d}$		

- Each part meets each part

Multiply out: four dot products

$$\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)\cdot\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)=R^2$$

parts of the right bracket

parts of the left bracket

	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$
$t\,\mathbf{d}$		

- Each part meets each part

Multiply out: four dot products

$$\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)\cdot\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)=R^2$$

parts of the right bracket

parts of the left bracket

	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$	

- Each part meets each part

Multiply out: four dot products

$$\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)\cdot\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)=R^2$$

parts of the right bracket

parts of the left bracket

	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$	$(t\,\mathbf{d})\cdot(t\,\mathbf{d})$

- Each part meets each part
- Four products, one per cell

Multiply out: four dot products

$$\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)\cdot\big((\mathbf{e}-\mathbf{c})+t\,\mathbf{d}\big)=R^2$$

parts of the right bracket

parts of the left bracket

	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$	$(t\,\mathbf{d})\cdot(t\,\mathbf{d})$

all four cells, added, $= R^2$

- Each part meets each part
- Four products, one per cell
- R^2 stays on the right

Pull t out; two cells match

parts of the right bracket

parts of the left bracket

	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$ no t in it	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$	$(t\,\mathbf{d})\cdot(t\,\mathbf{d})$

Pull t out; two cells match

parts of the left bracket \ parts of the right bracket	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$ no t in it	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$	$(t\,\mathbf{d})\cdot(t\,\mathbf{d})$

- A number slides out front

Pull t out; two cells match

parts of the left bracket \ parts of the right bracket	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$ no t in it	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$	$(t\,\mathbf{d})\cdot(t\,\mathbf{d})$

- A number slides out front
- In a dot, order does not matter

Pull t out; two cells match

parts of the right bracket

parts of the left bracket	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$ no t in it	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$	$(t\,\mathbf{d})\cdot(t\,\mathbf{d})$ $= t^2\,(\mathbf{d}\cdot\mathbf{d})$

- A number slides out front
- In a dot, order does not matter

Pull t out; two cells match

parts of the right bracket

parts of the left bracket

	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$ no t in it	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$	$(t\,\mathbf{d})\cdot(t\,\mathbf{d})$ $= t^2\,(\mathbf{d}\cdot\mathbf{d})$

- A number slides out front
- In a dot, order does not matter
- Two cells match: count twice

Pull t out; two cells match

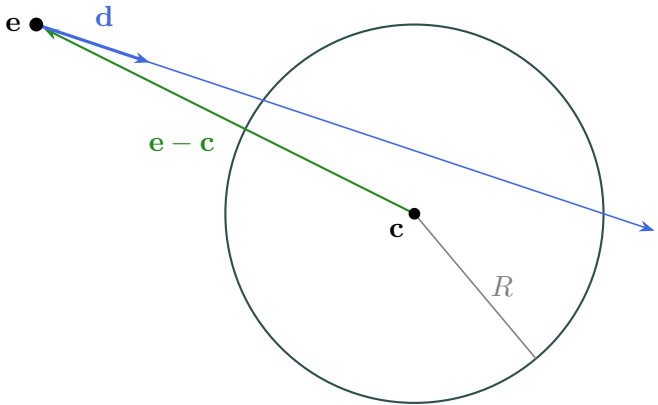
parts of the right bracket

parts of the left bracket	$(\mathbf{e}-\mathbf{c})$	$t\,\mathbf{d}$
$(\mathbf{e}-\mathbf{c})$	$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})$ no t in it	$(\mathbf{e}-\mathbf{c})\cdot(t\,\mathbf{d})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$
$t\,\mathbf{d}$	$(t\,\mathbf{d})\cdot(\mathbf{e}-\mathbf{c})$ $= t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$	$(t\,\mathbf{d})\cdot(t\,\mathbf{d})$ $= t^2\,(\mathbf{d}\cdot\mathbf{d})$

- A number slides out front
- In a dot, order does not matter
- Two cells match: count twice

$$(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c}) + 2\,t\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c}) + t^2\,(\mathbf{d}\cdot\mathbf{d}) = R^2$$

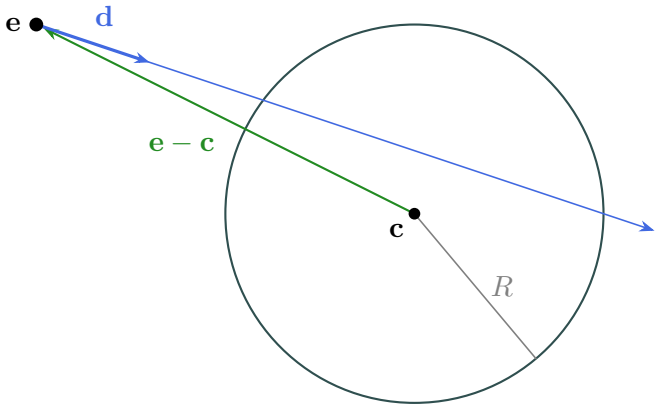
Sorted by powers of t: a quadratic



- Everything is known except t

$$(\mathbf{d}\cdot\mathbf{d})\,t^2 + 2\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})\,t + (\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c}) - R^2 = 0$$

Sorted by powers of t: a quadratic

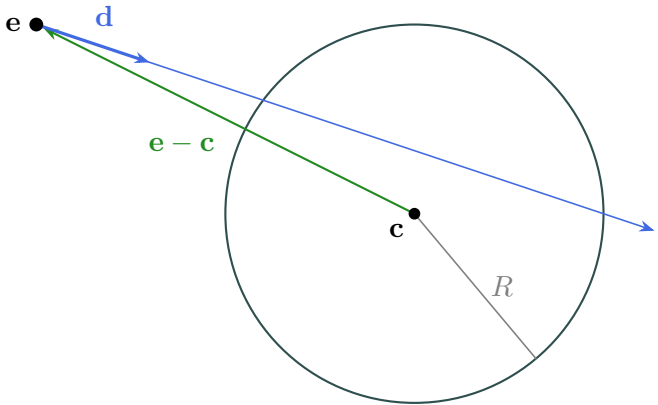


- Everything is known except t

$$A = \mathbf{d} \cdot \mathbf{d}$$

$$(\mathbf{d} \cdot \mathbf{d})\, t^2 + 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})\, t + (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2 = 0$$

Sorted by powers of t: a quadratic



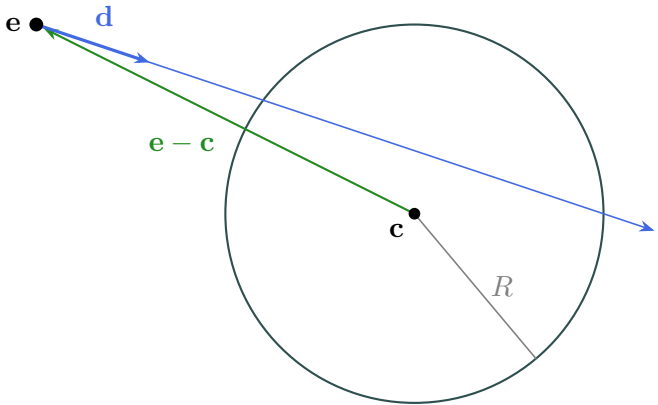
- Everything is known except t

$$A = \mathbf{d} \cdot \mathbf{d}$$

$$B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$$

$$(\mathbf{d} \cdot \mathbf{d})\,t^2 + 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})\,t + (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2 = 0$$

Sorted by powers of t: a quadratic



- Everything is known except t

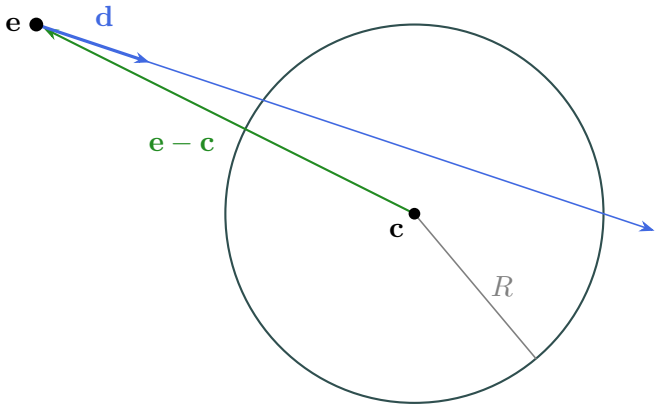
$$A = \mathbf{d} \cdot \mathbf{d}$$

$$B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$$

$$C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$$

$$(\mathbf{d} \cdot \mathbf{d})\, t^2 + 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})\, t + (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2 = 0$$

Sorted by powers of t: a quadratic



- Everything is known except t

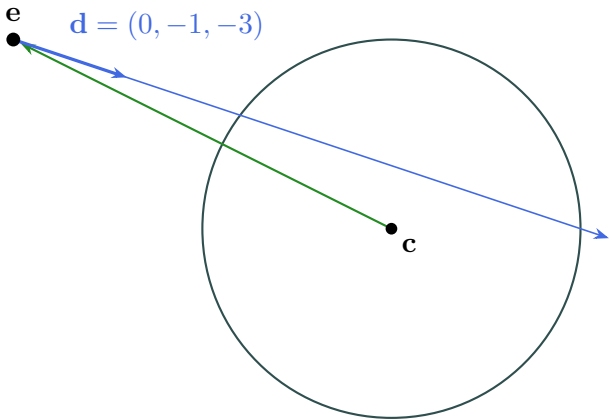
$$A = \mathbf{d} \cdot \mathbf{d}$$
$$B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$$
$$C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$$

$$A\,t^2 + B\,t + C = 0$$

- A classic quadratic in t

$$(\mathbf{d} \cdot \mathbf{d})\,t^2 + 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})\,t + (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2 = 0$$

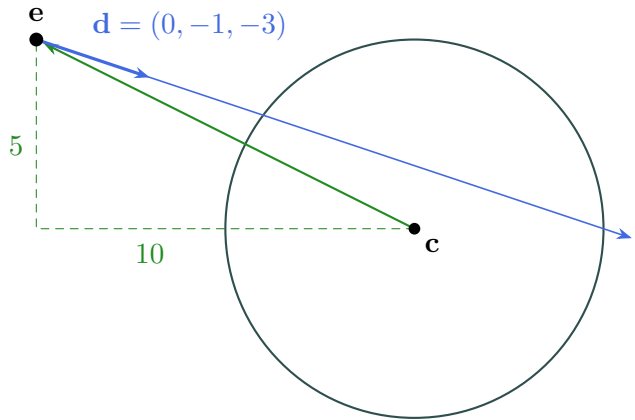
Worked: the arrow $\mathbf{e} - \mathbf{c}$, then A



- Try first: $\mathbf{e} - \mathbf{c}$, A, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$

$\mathbf{e} = (0, 5, 0), \quad \mathbf{c} = (0, 0, -10)$

Worked: the arrow $\mathbf{e} - \mathbf{c}$, then A



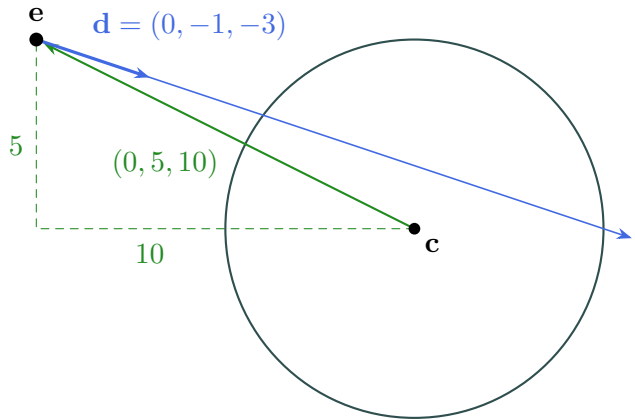
- First the arrow from $\mathbf{c}$ to $\mathbf{e}$

- Try first: $\mathbf{e} - \mathbf{c}$, A, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$

$\mathbf{e} = (0,\ 5,\ 0), \quad \mathbf{c} = (0,\ 0,\ -10)$

$\mathbf{e} - \mathbf{c} = (0, 5, 0) - (0, 0, -10)$

Worked: the arrow $\mathbf{e} - \mathbf{c}$, then A



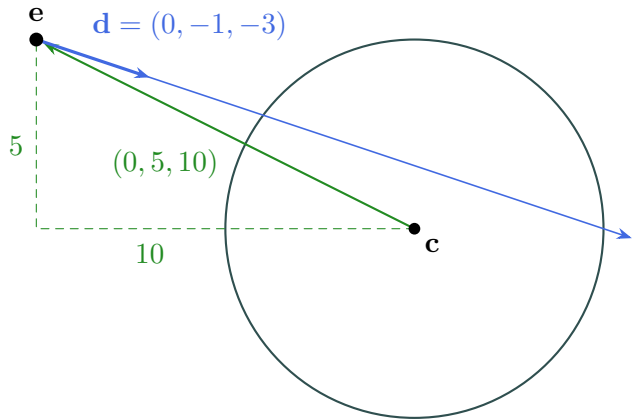
- First the arrow from $\mathbf{c}$ to $\mathbf{e}$

- Try first: $\mathbf{e} - \mathbf{c}$, A, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$

$$\mathbf{e} = (0,\ 5,\ 0), \quad \mathbf{c} = (0,\ 0,\ -10)$$

$$\begin{aligned} \mathbf{e} - \mathbf{c} &= (0, 5, 0) - (0, 0, -10) \\ &= (0,\ 5,\ 10) \end{aligned}$$

Worked: the arrow $\mathbf{e} - \mathbf{c}$, then A



- Try first: $\mathbf{e} - \mathbf{c}$, A, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$

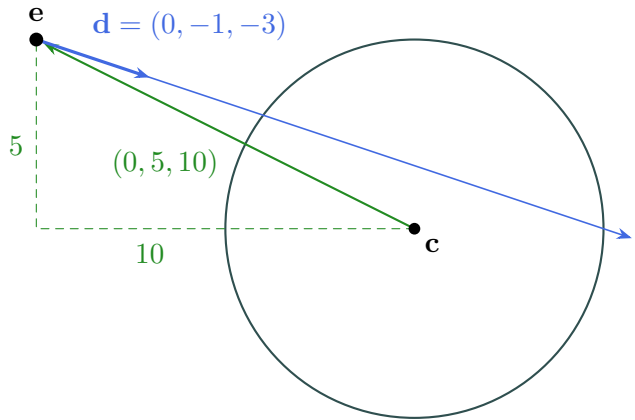
$$\mathbf{e} = (0,\ 5,\ 0), \quad \mathbf{c} = (0,\ 0,\ -10)$$

$$\mathbf{e} - \mathbf{c} = (0, 5, 0) - (0, 0, -10)$$
$$= (0,\ 5,\ 10)$$

$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + (-1)^2 + (-3)^2$$

- First the arrow from $\mathbf{c}$ to $\mathbf{e}$
- Then dot products, slot by slot

Worked: the arrow $\mathbf{e} - \mathbf{c}$, then A



- Try first: $\mathbf{e} - \mathbf{c}$, A, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$

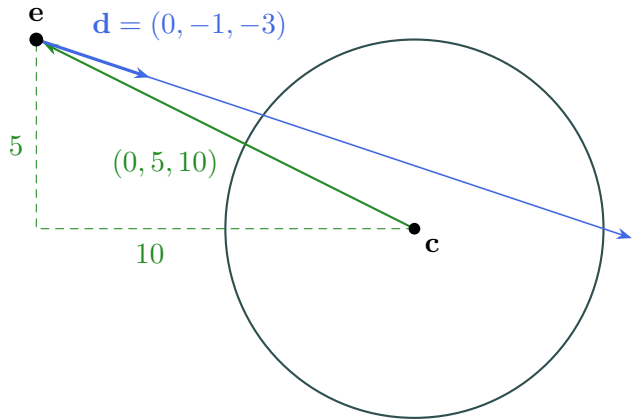
$$\mathbf{e} = (0,\ 5,\ 0), \quad \mathbf{c} = (0,\ 0,\ -10)$$

$$\begin{aligned} \mathbf{e} - \mathbf{c} &= (0, 5, 0) - (0, 0, -10) \\ &= (0,\ 5,\ 10) \end{aligned}$$

$$\begin{aligned} A = \mathbf{d} \cdot \mathbf{d} &= 0^2 + (-1)^2 + (-3)^2 \\ &= 0 + 1 + 9 = 10 \end{aligned}$$

- First the arrow from $\mathbf{c}$ to $\mathbf{e}$
- Then dot products, slot by slot

Worked: the arrow $\mathbf{e} - \mathbf{c}$, then A



- Try first: $\mathbf{e} - \mathbf{c}$, A, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$

$\mathbf{e} = (0,\ 5,\ 0), \quad \mathbf{c} = (0,\ 0,\ -10)$

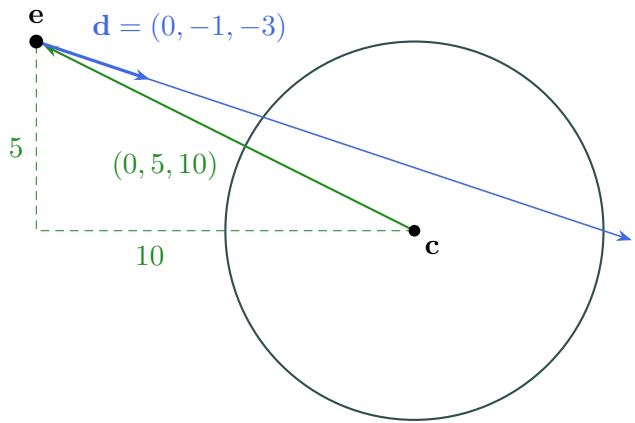
$$\mathbf{e} - \mathbf{c} = (0, 5, 0) - (0, 0, -10)$$
$$= (0,\ 5,\ 10)$$

$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + (-1)^2 + (-3)^2$$
$$= 0 + 1 + 9 = 10$$

$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (0, -1, -3) \cdot (0, 5, 10)$$

- First the arrow from $\mathbf{c}$ to $\mathbf{e}$
- Then dot products, slot by slot

Worked: the arrow $\mathbf{e} - \mathbf{c}$, then A



- Try first: $\mathbf{e} - \mathbf{c}$, A, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$

$\mathbf{e} = (0,\ 5,\ 0), \quad \mathbf{c} = (0,\ 0,\ -10)$

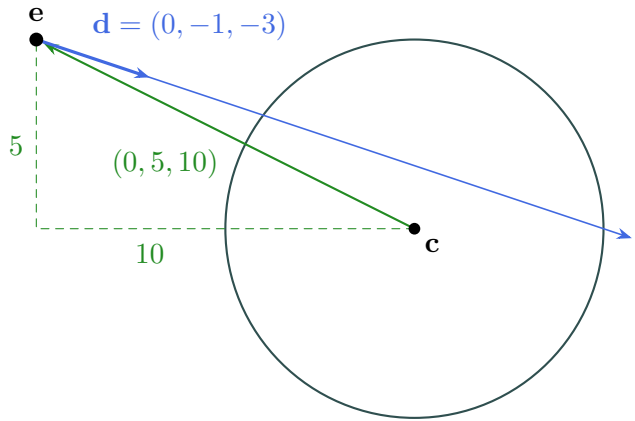
$$\mathbf{e} - \mathbf{c} = (0, 5, 0) - (0, 0, -10)$$
$$= (0,\ 5,\ 10)$$

$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + (-1)^2 + (-3)^2$$
$$= 0 + 1 + 9 = 10$$

$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (0, -1, -3) \cdot (0, 5, 10)$$
$$= (0)(0) + (-1)(5) + (-3)(10)$$

- First the arrow from $\mathbf{c}$ to $\mathbf{e}$
- Then dot products, slot by slot

Worked: the arrow $\mathbf{e} - \mathbf{c}$, then A



- Try first: $\mathbf{e} - \mathbf{c}$, A, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$

$\mathbf{e} = (0,\ 5,\ 0), \quad \mathbf{c} = (0,\ 0,\ -10)$

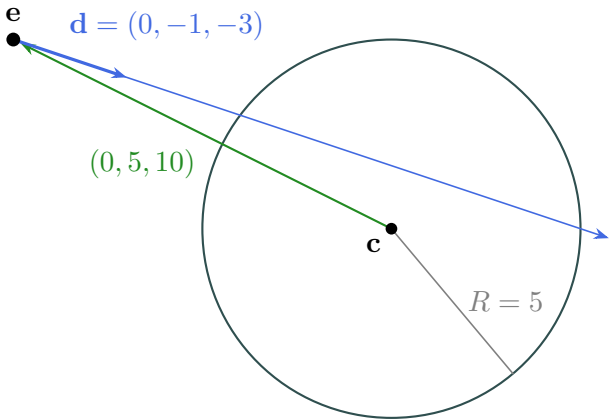
$$\mathbf{e} - \mathbf{c} = (0, 5, 0) - (0, 0, -10)$$
$$= (0,\ 5,\ 10)$$

$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + (-1)^2 + (-3)^2$$
$$= 0 + 1 + 9 = 10$$

$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (0, -1, -3) \cdot (0, 5, 10)$$
$$= (0)(0) + (-1)(5) + (-3)(10)$$
$$= 0 - 5 - 30 = -35$$

- First the arrow from $\mathbf{c}$ to $\mathbf{e}$
- Then dot products, slot by slot

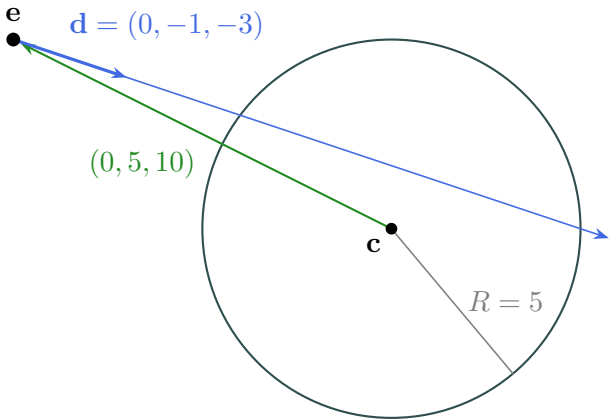
Worked: B, C, and our quadratic



- Try first: B, C, then the quadratic

$$A = 10, \quad \mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = -35$$

Worked: B, C, and our quadratic

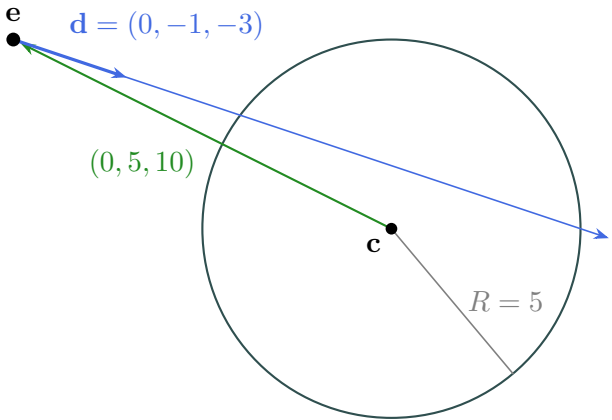


- Try first: B, C, then the quadratic

$$A = 10, \quad \mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = -35$$

$$B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = 2\,(-35) = -70$$

Worked: B, C, and our quadratic



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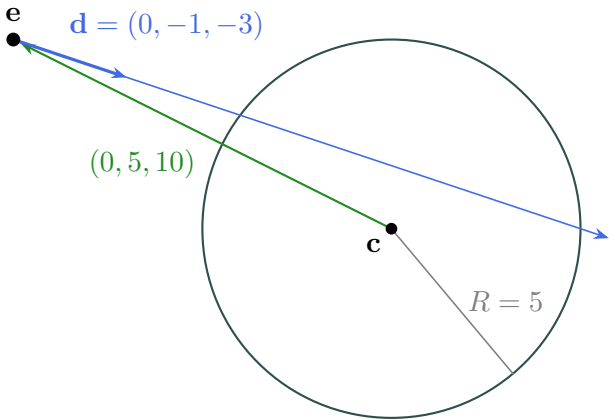
Worked: B, C, and our quadratic



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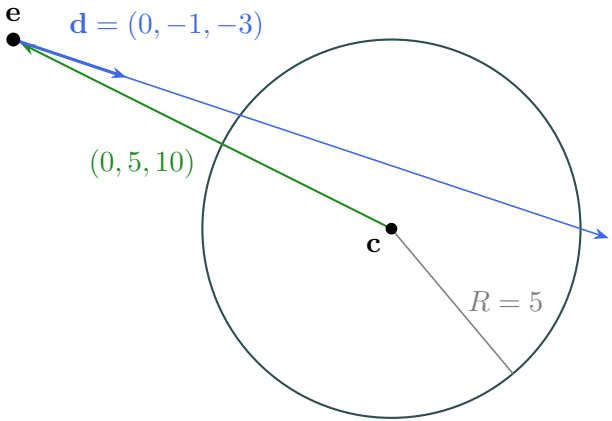
$$(\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) = 0^2 + 5^2 + 10^2$$
$$= 0 + 25 + 100 = 125$$

Worked: B, C, and our quadratic



- Try first: B, C, then the quadratic

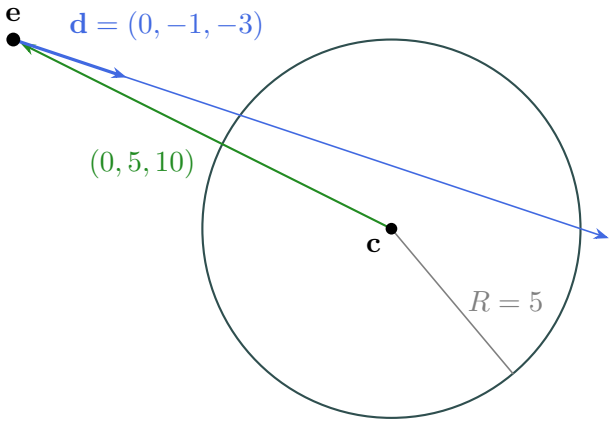
$$A = 10, \quad \mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = -35$$

$$B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = 2\,(-35) = -70$$

$$(\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) = 0^2 + 5^2 + 10^2$$
$$= 0 + 25 + 100 = 125$$

$$C = 125 - R^2 = 125 - 25 = 100$$

Worked: B, C, and our quadratic



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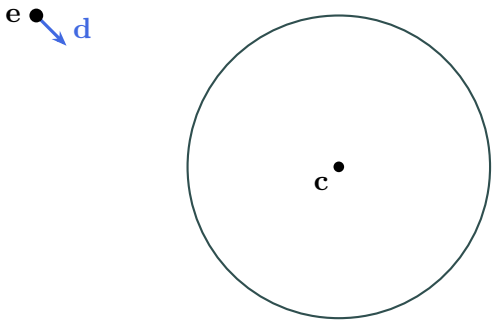
$$(\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) = 0^2 + 5^2 + 10^2$$
$$= 0 + 25 + 100 = 125$$

$$C = 125 - R^2 = 125 - 25 = 100$$

$$\boxed{10\,t^2 - 70\,t + 100 = 0}$$

- A, B, C: plain numbers now
- Only t is still unknown

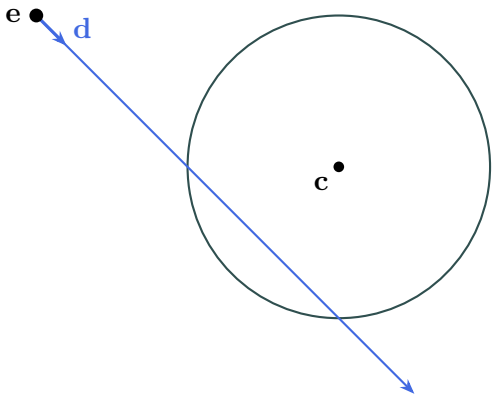
Quick check 1: the quadratic for a steeper ray



$$\mathbf{e} = (0, 5, 0), \quad \mathbf{d} = (0, -1, -1)$$
$$\mathbf{c} = (0, 0, -10), \quad R = 5$$

- Find A, B and C
- Then write the ray's quadratic

Quick check 1: the quadratic for a steeper ray



$\mathbf{e} = (0, 5, 0), \quad \mathbf{d} = (0, -1, -1)$

$\mathbf{c} = (0, 0, -10), \quad R = 5$

- Find A, B and C
- Then write the ray's quadratic

$A = 0 + 1 + 1 = 2$

$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = 0 - 5 - 10 = -15, \quad B = -30$

$C = 100$ (as before)

$$2\,t^2 - 30\,t + 100 = 0$$

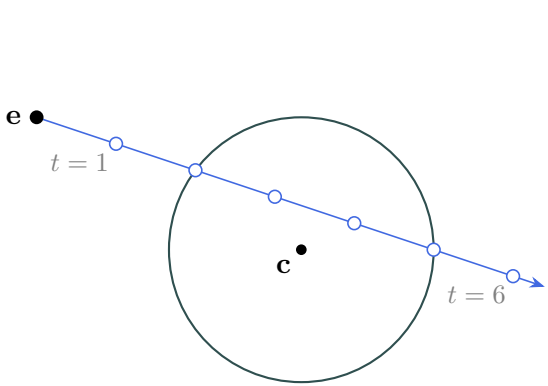
Recap: the ray in the sphere's equation

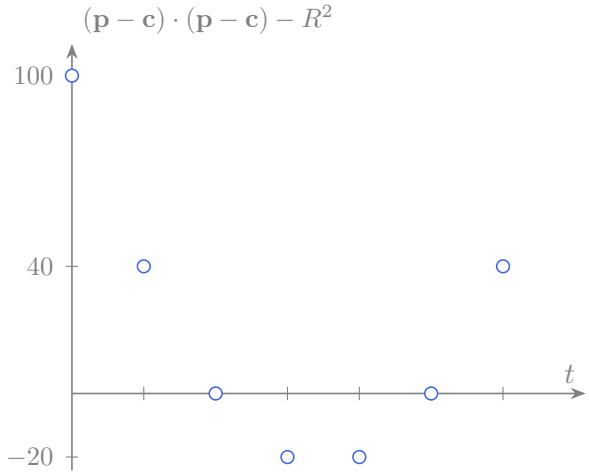
$$\underbrace{(\mathbf{d}\cdot\mathbf{d})}_{A}\,t^2 \;+\; \underbrace{2\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})}_{B}\,t \;+\; \underbrace{(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})-R^2}_{C} \;=\; 0$$

Solving for t

Miss, graze or hit, and which root counts

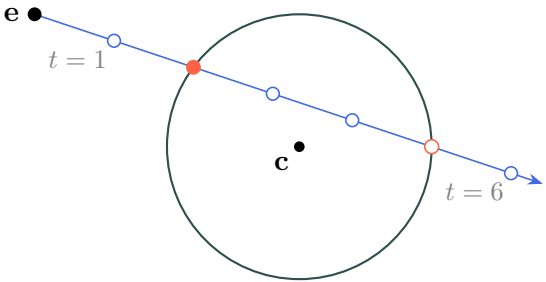
Zero crossings: where the ray meets the sphere

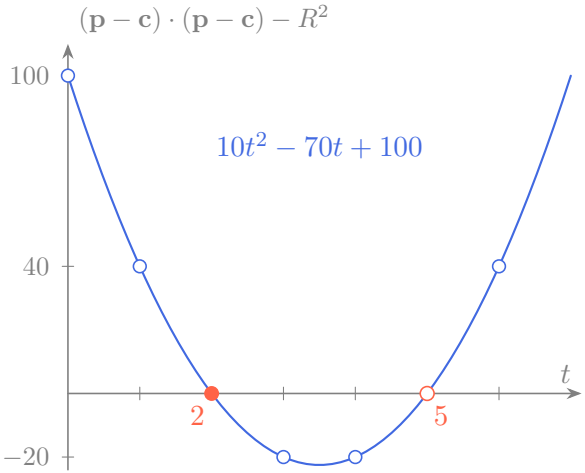




- Height: squared distance minus R^2

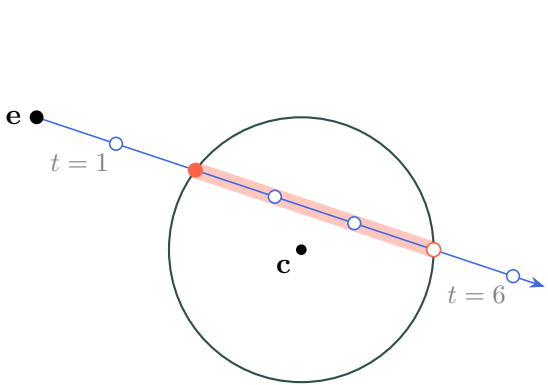
Zero crossings: where the ray meets the sphere

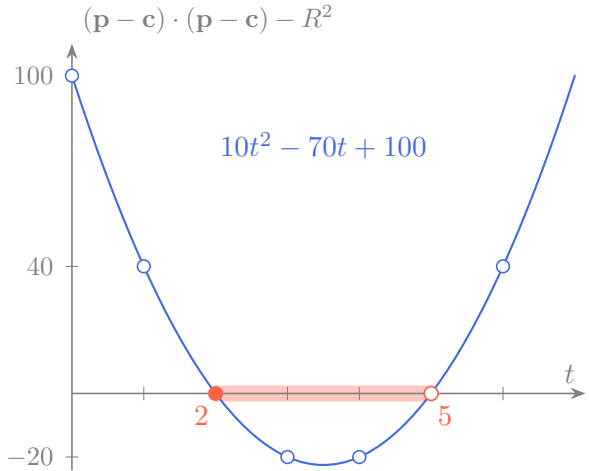




- Height: squared distance minus R^2
- The dots lie on our quadratic

Zero crossings: where the ray meets the sphere

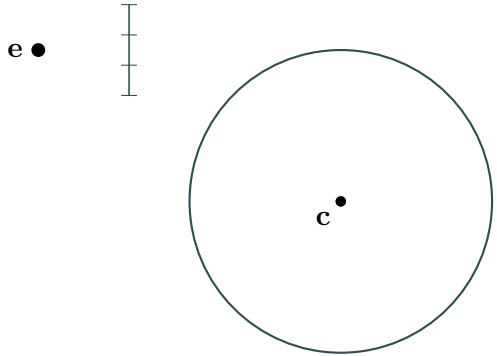




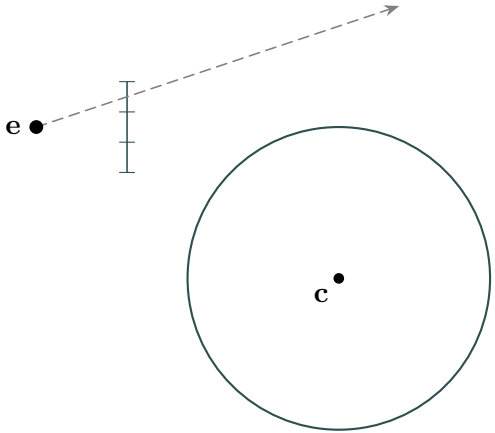
- Height: squared distance minus R^2
- The dots lie on our quadratic
- Below zero: inside the sphere

The sign of D: miss, graze, or hit

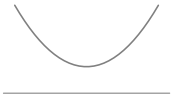
$$t = \frac{-B \pm \sqrt{D}}{2A}, \qquad D = B^2 - 4AC$$



The sign of D: miss, graze, or hit

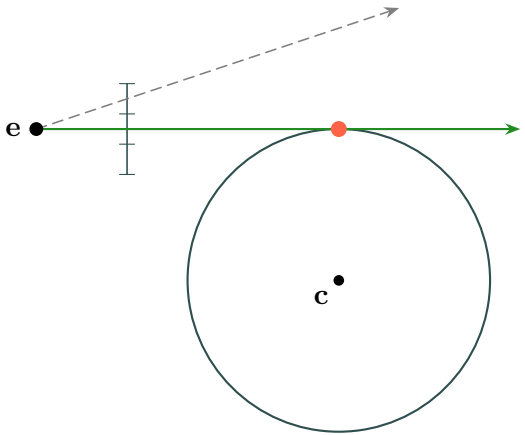


$$t = \frac{-B \pm \sqrt{D}}{2A}, \qquad D = B^2 - 4AC$$



- $D < 0$: the ray misses

The sign of D: miss, graze, or hit



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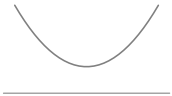

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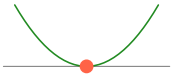
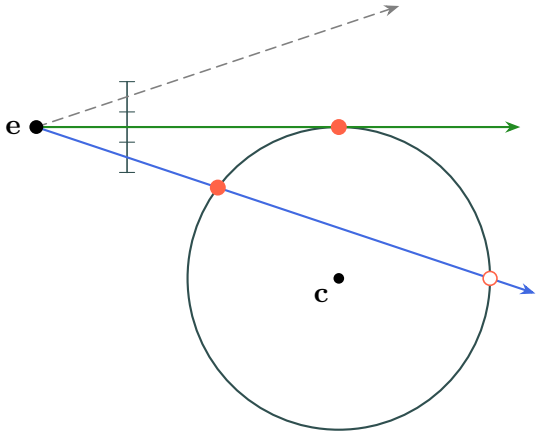
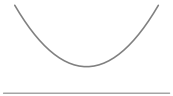


- $D = 0$: grazes at one point

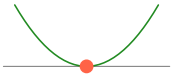
The sign of D: miss, graze, or hit



$$t = \frac{-B \pm \sqrt{D}}{2A}, \qquad D = B^2 - 4AC$$



- $D < 0$: the ray misses



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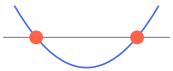
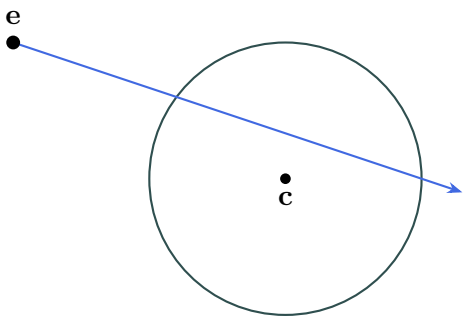


- $D > 0$: goes in, then out

Worked: where our ray goes in and out

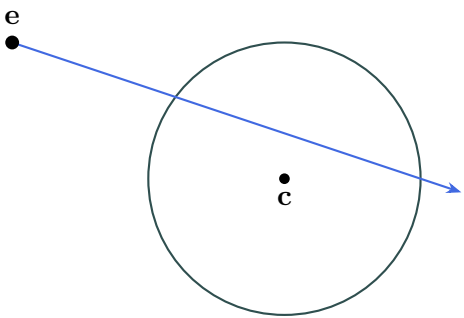


- Try first: D, t_-, t_+, the two points

$$10\,t^2 - 70\,t + 100 = 0$$

$$\mathbf{e} = (0,\, 5,\, 0), \quad \mathbf{d} = (0,\, -1,\, -3)$$

Worked: where our ray goes in and out



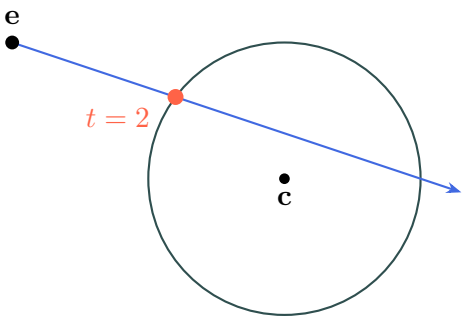
- Try first: D, t_-, t_+, the two points

$$10\,t^2 - 70\,t + 100 = 0$$

$$\mathbf{e} = (0,\, 5,\, 0), \quad \mathbf{d} = (0,\, -1,\, -3)$$

$$D = (-70)^2 - 4 \cdot 10 \cdot 100 = 900$$

Worked: where our ray goes in and out



- Try first: D, t_-, t_+, the two points

$$10\,t^2 - 70\,t + 100 = 0$$

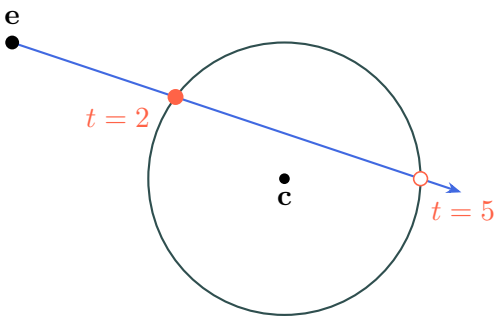
$$\mathbf{e} = (0,\, 5,\, 0), \quad \mathbf{d} = (0,\, -1,\, -3)$$

$$D = (-70)^2 - 4 \cdot 10 \cdot 100 = 900$$

$$t_- = \frac{-B - \sqrt{D}}{2A} = \frac{70 - 30}{20} = 2$$

- Minus sign: the way in

Worked: where our ray goes in and out



- Minus sign: the way in
- Plus sign: the way out

- Try first: D, t_-, t_+, the two points

$$10\,t^2 - 70\,t + 100 = 0$$

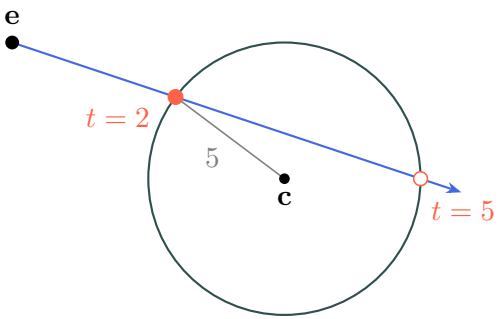
$$\mathbf{e} = (0,\, 5,\, 0), \quad \mathbf{d} = (0,\, -1,\, -3)$$

$$D = (-70)^2 - 4 \cdot 10 \cdot 100 = 900$$

$$t_- = \frac{-B - \sqrt{D}}{2A} = \frac{70 - 30}{20} = 2$$

$$t_+ = \frac{-B + \sqrt{D}}{2A} = \frac{70 + 30}{20} = 5$$

Worked: where our ray goes in and out



- Minus sign: the way in
- Plus sign: the way out

- Try first: D, t_-, t_+, the two points

$$10\,t^2 - 70\,t + 100 = 0$$

$$\mathbf{e} = (0,\,5,\,0), \quad \mathbf{d} = (0,\,-1,\,-3)$$

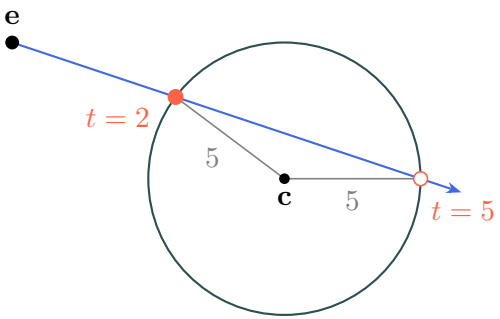
$$D = (-70)^2 - 4 \cdot 10 \cdot 100 = 900$$

$$t_- = \frac{-B - \sqrt{D}}{2A} = \frac{70 - 30}{20} = 2$$

$$t_+ = \frac{-B + \sqrt{D}}{2A} = \frac{70 + 30}{20} = 5$$

$$\mathbf{p}(2) = (0,5,0) + 2\,(0,-1,-3) = (0,3,-6)$$

Worked: where our ray goes in and out



- Minus sign: the way in
- Plus sign: the way out

- Try first: D, t_-, t_+, the two points

$$10\,t^2 - 70\,t + 100 = 0$$

$$\mathbf{e} = (0,\,5,\,0), \quad \mathbf{d} = (0,\,-1,\,-3)$$

$$D = (-70)^2 - 4 \cdot 10 \cdot 100 = 900$$

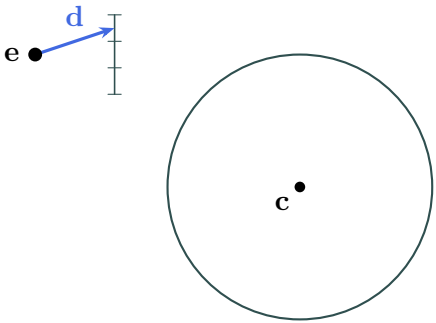
$$t_- = \frac{-B - \sqrt{D}}{2A} = \frac{70 - 30}{20} = 2$$

$$t_+ = \frac{-B + \sqrt{D}}{2A} = \frac{70 + 30}{20} = 5$$

$$\mathbf{p}(2) = (0,5,0) + 2\,(0,-1,-3) = (0,3,-6)$$

$$\mathbf{p}(5) = (0,5,0) + 5\,(0,-1,-3) = (0,0,-15)$$

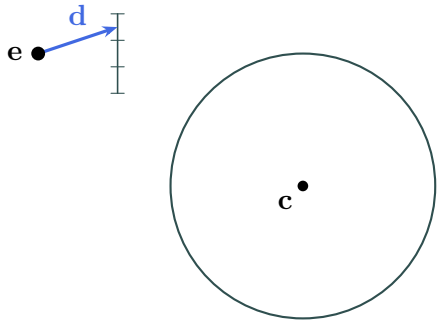
Worked: the ray through the top pixel



- Try first: A, B, C, then D. Hit or miss?

$$\mathbf{d} = (0,\ 1,\ -3), \quad \mathbf{e} - \mathbf{c} = (0,\ 5,\ 10), \quad R = 5$$

Worked: the ray through the top pixel

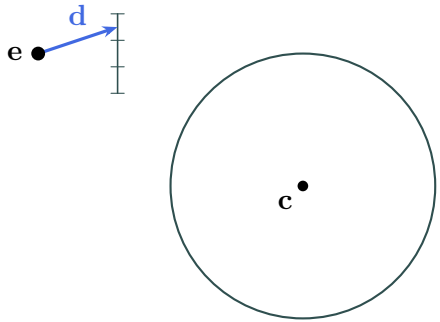


- Try first: A, B, C, then D. Hit or miss?

$$\mathbf{d} = (0,\ 1,\ -3), \quad \mathbf{e} - \mathbf{c} = (0,\ 5,\ 10), \quad R = 5$$

$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + 1^2 + (-3)^2 = 10$$

Worked: the ray through the top pixel



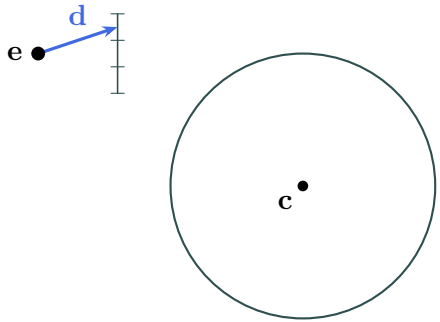
- Try first: A, B, C, then D. Hit or miss?

$$\mathbf{d} = (0,\ 1,\ -3), \quad \mathbf{e} - \mathbf{c} = (0,\ 5,\ 10), \quad R = 5$$

$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + 1^2 + (-3)^2 = 10$$

$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (0)(0) + (1)(5) + (-3)(10)$$
$$= 0 + 5 - 30 = -25$$

Worked: the ray through the top pixel



- Try first: A, B, C, then D. Hit or miss?

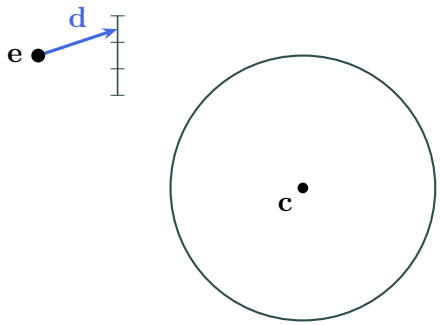
$$\mathbf{d} = (0,\ 1,\ -3), \quad \mathbf{e} - \mathbf{c} = (0,\ 5,\ 10), \quad R = 5$$

$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + 1^2 + (-3)^2 = 10$$

$$\begin{aligned} \mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) &= (0)(0) + (1)(5) + (-3)(10) \\ &= 0 + 5 - 30 = -25 \end{aligned}$$

$$B = 2\,(-25) = -50$$

Worked: the ray through the top pixel



- Same $\mathbf{e}$ and sphere: C stays 100

- Try first: A, B, C, then D. Hit or miss?

$$\mathbf{d} = (0,\ 1,\ -3), \quad \mathbf{e} - \mathbf{c} = (0,\ 5,\ 10), \quad R = 5$$

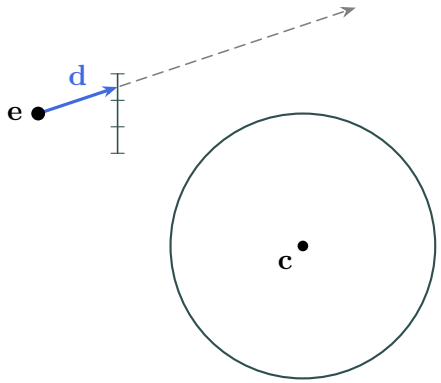
$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + 1^2 + (-3)^2 = 10$$

$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (0)(0) + (1)(5) + (-3)(10)$$
$$= 0 + 5 - 30 = -25$$

$$B = 2\,(-25) = -50$$

$$C = 125 - 25 = 100 \quad \text{(no } \mathbf{d} \text{ in it)}$$

Worked: the ray through the top pixel



- Same $\mathbf{e}$ and sphere: C stays 100
- $D < 0$: no crossing, a miss

- Try first: A, B, C, then D. Hit or miss?

$$\mathbf{d} = (0,\ 1,\ -3), \quad \mathbf{e} - \mathbf{c} = (0,\ 5,\ 10), \quad R = 5$$

$$A = \mathbf{d} \cdot \mathbf{d} = 0^2 + 1^2 + (-3)^2 = 10$$

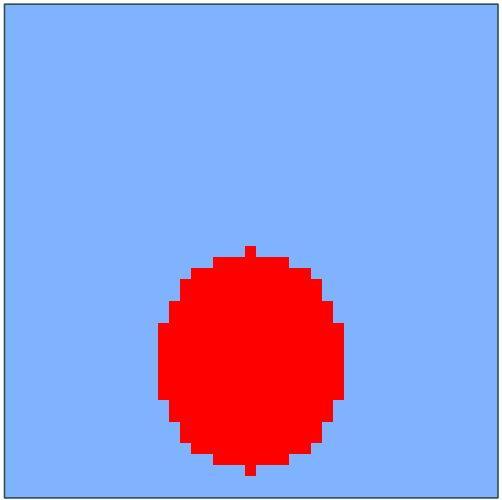
$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (0)(0) + (1)(5) + (-3)(10)$$
$$= 0 + 5 - 30 = -25$$

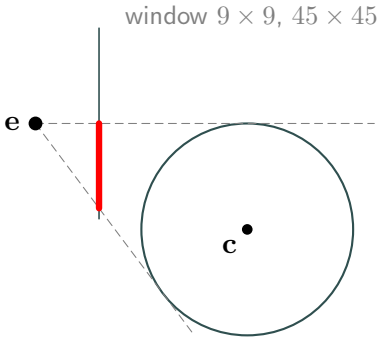
$$B = 2\,(-25) = -50$$

$$C = 125 - 25 = 100 \quad \text{(no } \mathbf{d} \text{ in it)}$$

$$D = B^2 - 4AC = (-50)^2 - 4 \cdot 10 \cdot 100$$
$$= 2500 - 4000 = -1500$$

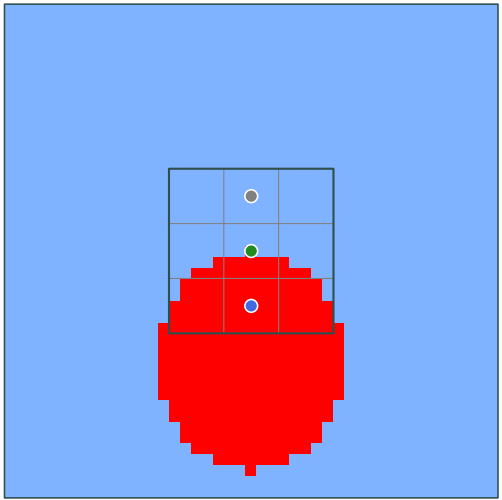
Our first picture: red where the ray hits

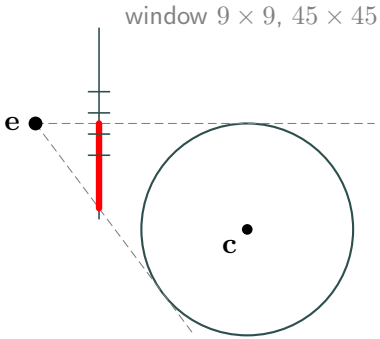




- One ray through every pixel
- $D \geq 0$: paint it red, else sky

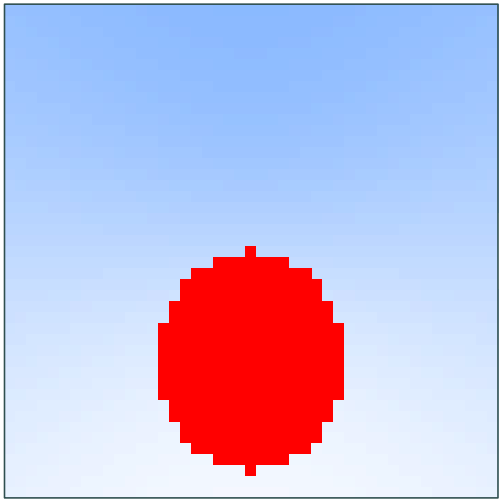
Our first picture: red where the ray hits





- One ray through every pixel
- $D \geq 0$: paint it red, else sky
- Middle: our 3×3 window

Misses show the sky: white blended into blue

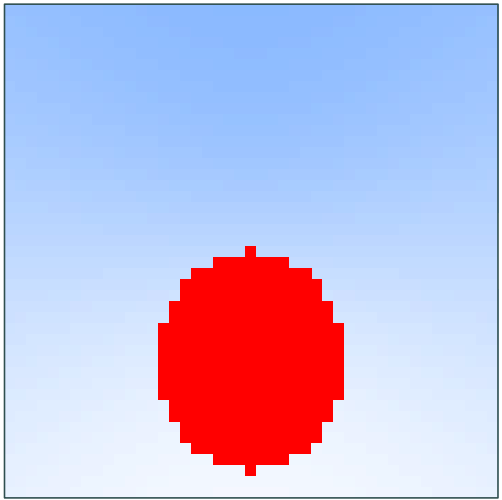


$$a = \frac{1}{2}(\hat{y} + 1)$$

$\hat{y}$: the height of $\mathbf{d}$ scaled to length 1

- Blend by the ray's height

Misses show the sky: white blended into blue



$a = \frac{1}{2}(\hat{y} + 1)$

$\hat{y}$: the height of $\mathbf{d}$ scaled to length 1

colour $= (1 - a)\,(1,\ 1,\ 1) + a\,(0.5,\ 0.7,\ 1)$

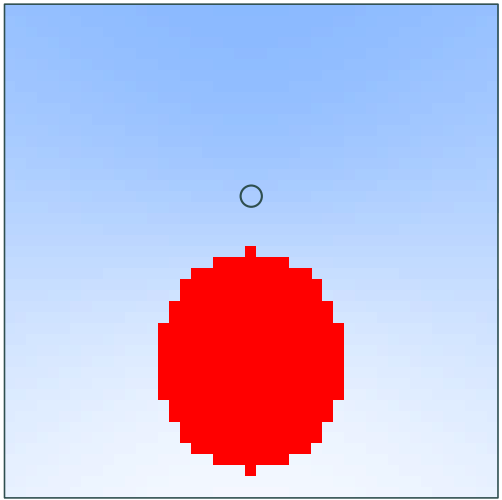


$a = 0$: white straight down

$a = 1$: blue straight up

- Blend by the ray's height
- A lerp, as in lecture 1

Misses show the sky: white blended into blue



$a = \frac{1}{2}(\hat{y} + 1)$

$\hat{y}$: the height of $\mathbf{d}$ scaled to length 1

$$\text{colour} = (1 - a)\,(1,\ 1,\ 1) + a\,(0.5,\ 0.7,\ 1)$$

$a = 0.66$

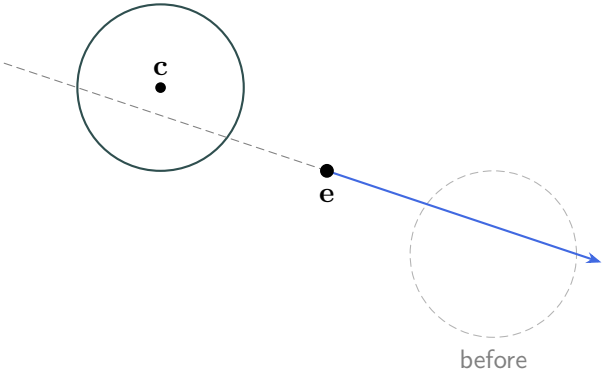


$a = 0$: white straight down

$a = 1$: blue straight up

- Blend by the ray's height
- A lerp, as in lecture 1
- Top pixel's ray: $(0.67,\ 0.80,\ 1)$

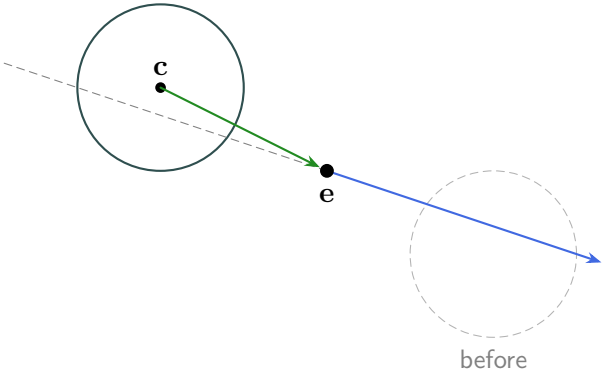
A sphere behind the eye: the same picture



$\mathbf{c} = (0, 10, 10)$

moved behind the eye

A sphere behind the eye: the same picture



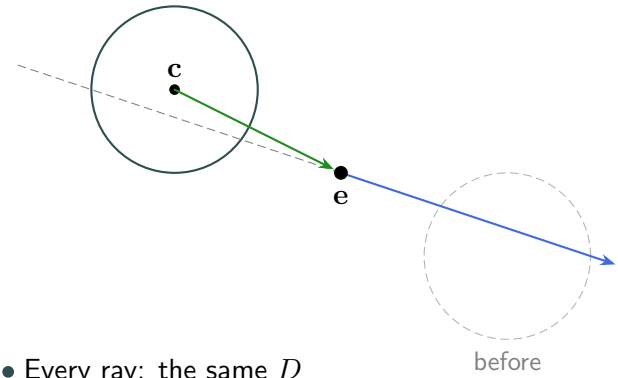
$\mathbf{c} = (0,\ 10,\ 10)$

moved behind the eye

$\mathbf{e} - \mathbf{c} = (0,\ -5,\ -10)$

$B = 2\,(35) = 70, \quad C = 100$

A sphere behind the eye: the same picture



$\mathbf{c} = (0,\ 10,\ 10)$

moved behind the eye

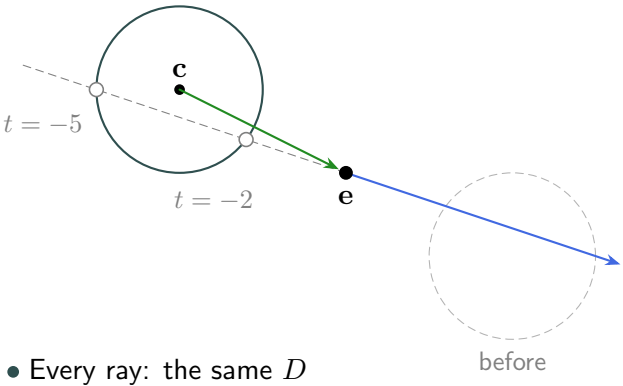
$\mathbf{e} - \mathbf{c} = (0,\ -5,\ -10)$

$B = 2\,(35) = 70, \quad C = 100$

$D = 70^2 - 4 \cdot 10 \cdot 100 = 900$

- Every ray: the same D

A sphere behind the eye: the same picture



$\mathbf{c} = (0,\ 10,\ 10)$

moved behind the eye

$\mathbf{e} - \mathbf{c} = (0,\ -5,\ -10)$

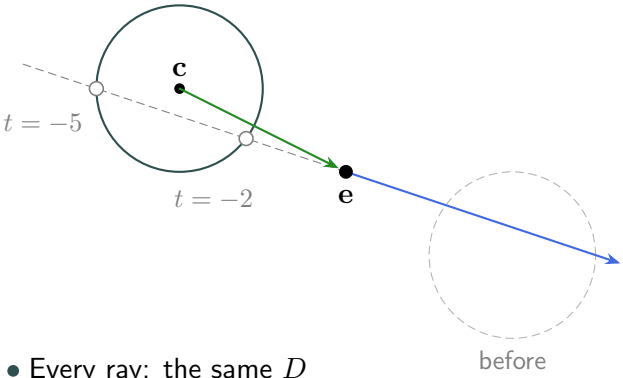
$B = 2\,(35) = 70, \quad C = 100$

$D = 70^2 - 4 \cdot 10 \cdot 100 = 900$

$$t = \frac{-70 \pm 30}{20} = -5,\ -2$$

- Every ray: the same D
- Both crossings behind the eye

A sphere behind the eye: the same picture



$\mathbf{c} = (0,\ 10,\ 10)$

moved behind the eye

$\mathbf{e} - \mathbf{c} = (0,\ -5,\ -10)$

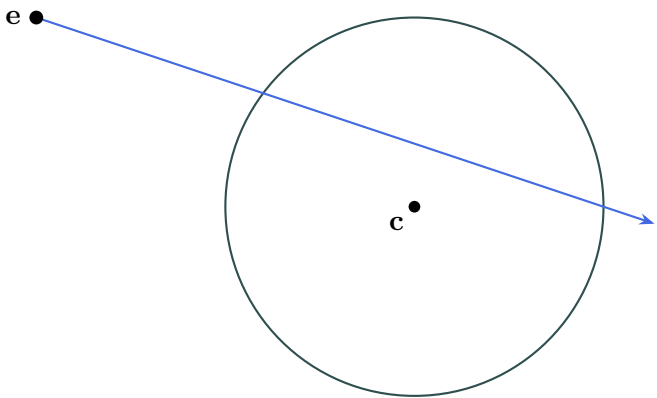
$B = 2\,(35) = 70, \quad C = 100$

$D = 70^2 - 4 \cdot 10 \cdot 100 = 900$

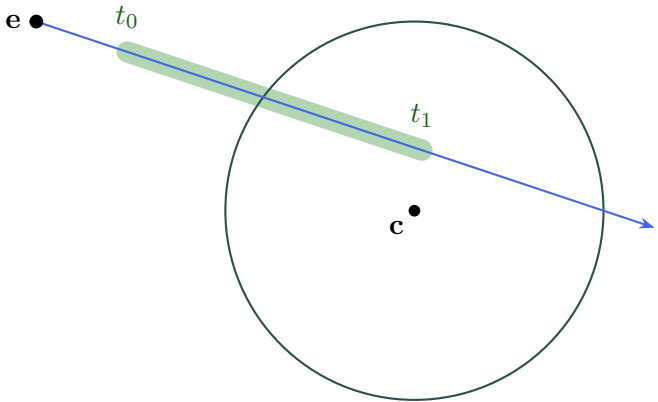
$$t = \frac{-70 \pm 30}{20} = -5,\ -2$$

- Every ray: the same D
- Both crossings behind the eye
- Fix: count only $t \geq 0$

Only hits with t in $[t_0, t_1]$ count

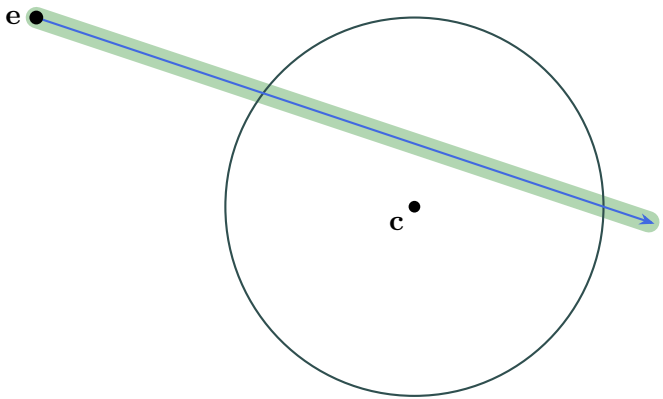


Only hits with t in $[t_0, t_1]$ count



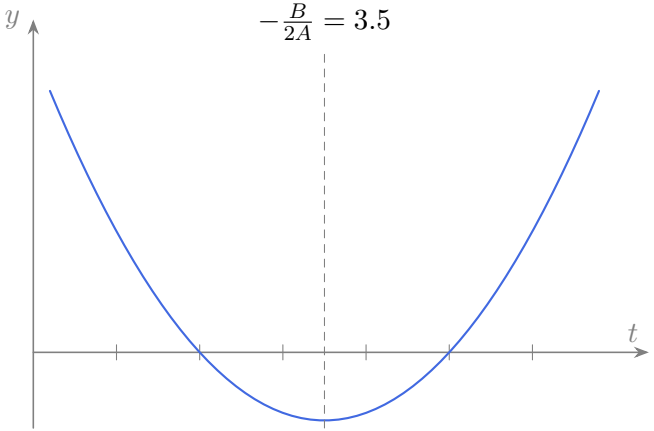
- First hit with t in $[t_0, t_1]$

Only hits with t in $[t_0, t_1]$ count



- First hit with t in $[t_0, t_1]$
- Basic case: $t_0 = 0$, $t_1 = +\infty$
- Everything ahead of the eye

The minus sign gives the smaller root

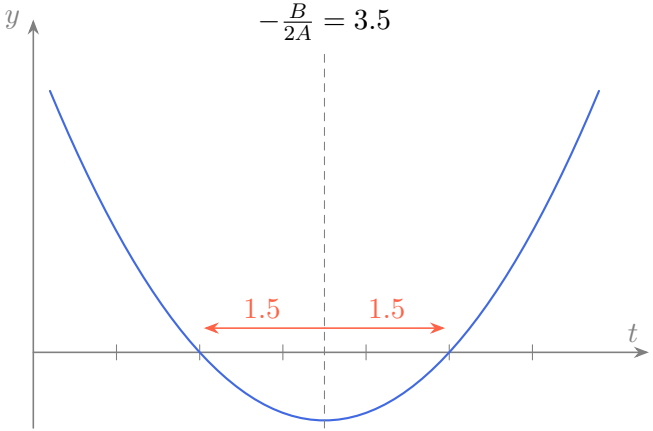


$$10\,t^2 - 70\,t + 100 = 0$$

$$t_- = \frac{-B - \sqrt{D}}{2A}$$

$$t_+ = \frac{-B + \sqrt{D}}{2A}$$

The minus sign gives the smaller root



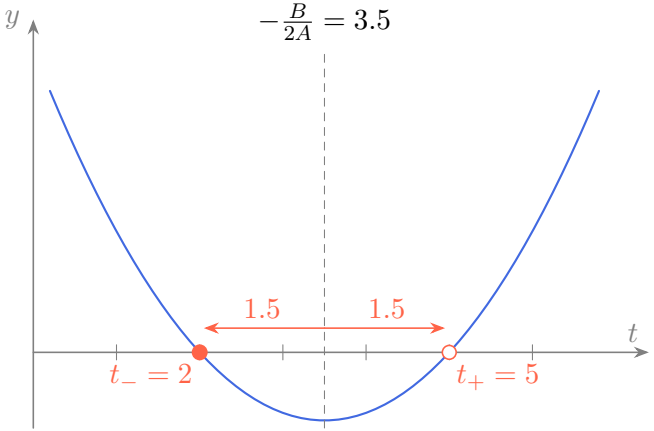
$$10\,t^2 - 70\,t + 100 = 0$$

$$t_- = \frac{-B - \sqrt{D}}{2A}$$

$$t_+ = \frac{-B + \sqrt{D}}{2A}$$

- Centre 3.5, step 1.5 each way

The minus sign gives the smaller root



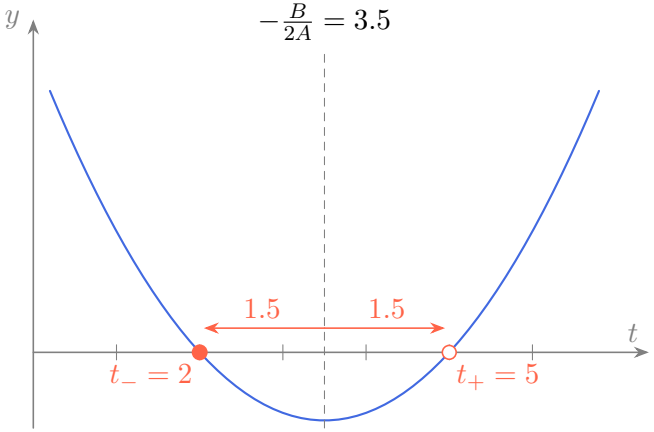
$$10\,t^2 - 70\,t + 100 = 0$$

$$t_- = \frac{-B - \sqrt{D}}{2A} = 3.5 - 1.5 = 2$$

$$t_+ = \frac{-B + \sqrt{D}}{2A} = 3.5 + 1.5 = 5$$

- Centre 3.5, step 1.5 each way

The minus sign gives the smaller root



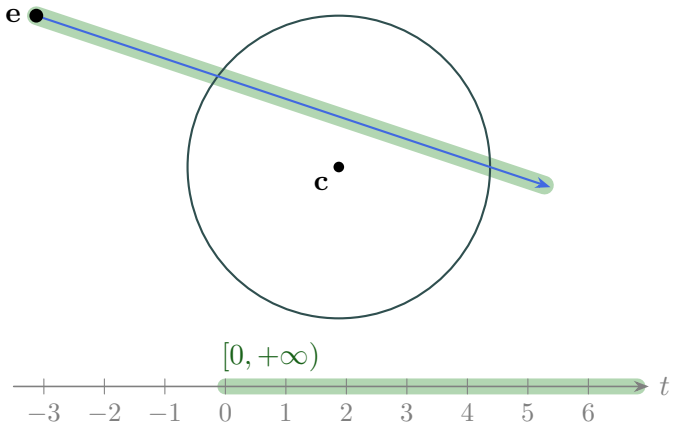
$$10\,t^2 - 70\,t + 100 = 0$$

$$t_- = \frac{-B - \sqrt{D}}{2A} = 3.5 - 1.5 = 2$$

$$t_+ = \frac{-B + \sqrt{D}}{2A} = 3.5 + 1.5 = 5$$

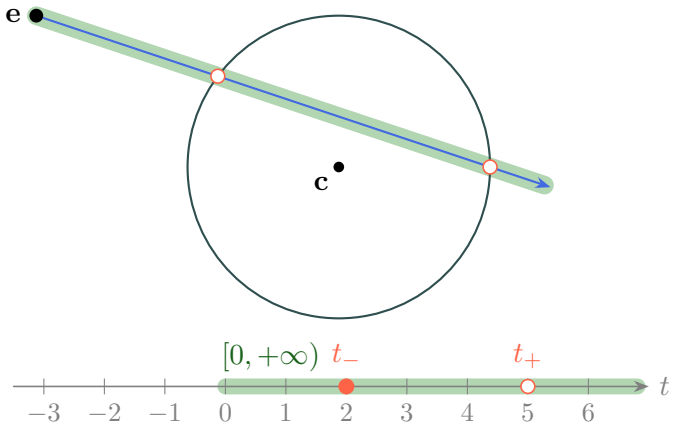
- Centre 3.5, step 1.5 each way
- $A = \mathbf{d} \cdot \mathbf{d}$ is always positive
- So $t_- \leq t_+$ for every ray

Case 1: smaller root inside, so first hit



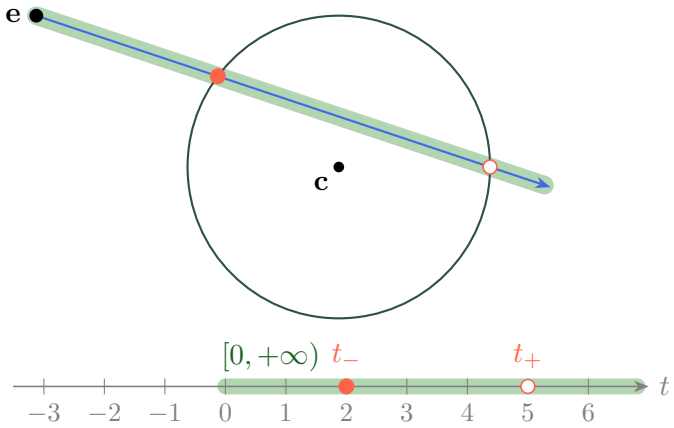
- Basic interval: $[0, +\infty)$

Case 1: smaller root inside, so first hit



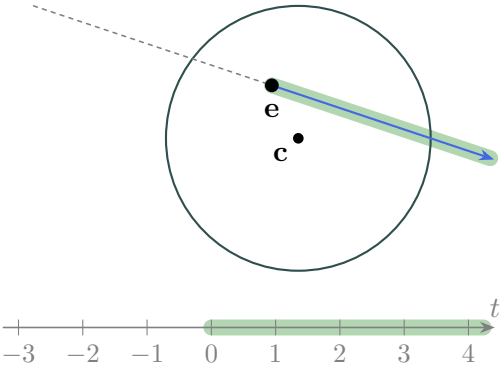
- Basic interval: $[0, +\infty)$
- Smaller root $t_- = 2$ is inside

Case 1: smaller root inside, so first hit



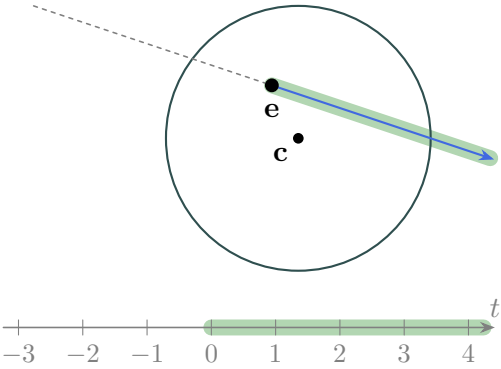
- Basic interval: $[0, +\infty)$
- Smaller root $t_- = 2$ is inside
- So t_- is the first hit

Case 2: start inside, take the larger root



$$\mathbf{e} = (0, 2, -9), \quad \mathbf{e} - \mathbf{c} = (0, 2, 1)$$

Case 2: start inside, take the larger root



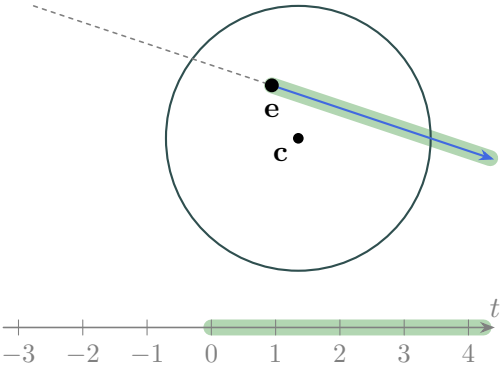
$$\mathbf{e} = (0, 2, -9), \quad \mathbf{e} - \mathbf{c} = (0, 2, 1)$$

$$A = \mathbf{d} \cdot \mathbf{d} = 10$$

$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (-1)(2) + (-3)(1) = -5$$

$$B = 2\,(-5) = -10$$

Case 2: start inside, take the larger root



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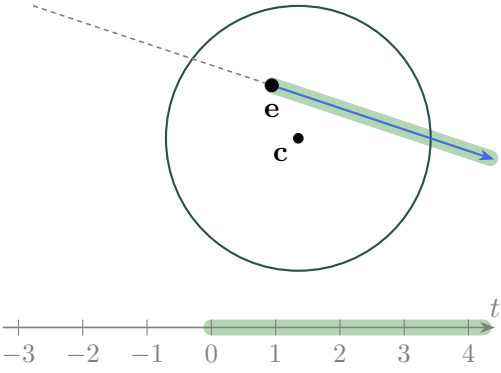
$$B = 2\,(-5) = -10$$

$$C = (0 + 4 + 1) - 25 = -20$$

Case 2: start inside, take the larger root



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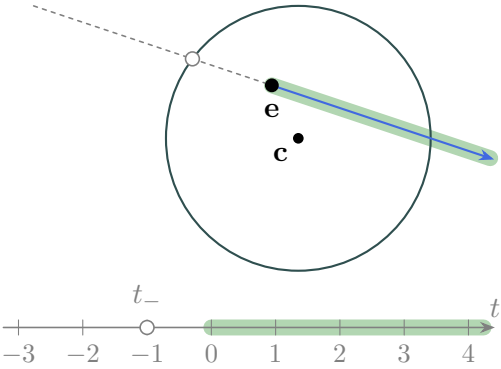
$$B = 2\,(-5) = -10$$

$$C = (0 + 4 + 1) - 25 = -20$$

$$D = (-10)^2 - 4 \cdot 10 \cdot (-20) = 100 + 800 = 900$$

$$t = \frac{10 \pm 30}{20} = -1 \quad \text{or} \quad 2$$

Case 2: start inside, take the larger root



- $t_- = -1$: behind the start

$$\mathbf{e} = (0, 2, -9), \quad \mathbf{e} - \mathbf{c} = (0, 2, 1)$$

$$A = \mathbf{d} \cdot \mathbf{d} = 10$$

$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (-1)(2) + (-3)(1) = -5$$

$$B = 2\,(-5) = -10$$

$$C = (0 + 4 + 1) - 25 = -20$$

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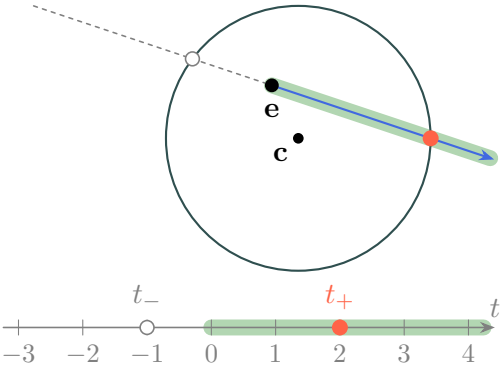
Case 2: start inside, take the larger root



- $t_- = -1$: behind the start
- $t_+ = 2$ is inside: the hit

$$\mathbf{e} = (0, 2, -9), \quad \mathbf{e} - \mathbf{c} = (0, 2, 1)$$

$$A = \mathbf{d} \cdot \mathbf{d} = 10$$

$$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = (-1)(2) + (-3)(1) = -5$$

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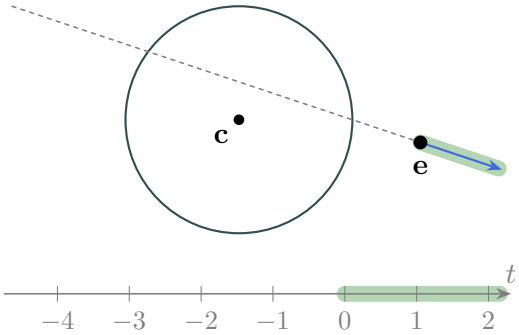
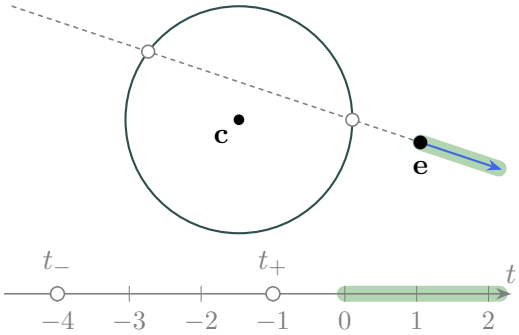
Case 3: neither root inside: no hit



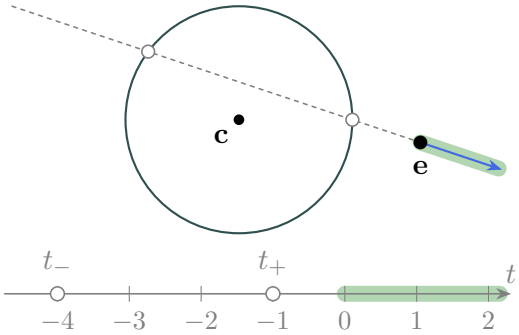
Case 3: neither root inside: no hit

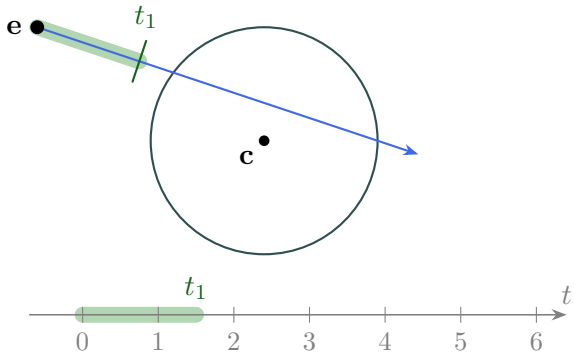


$10t^2 + 50t + 40 = 0$: $t = -4,\ -1$

- Sphere behind: both roots below t_0

Case 3: neither root inside: no hit





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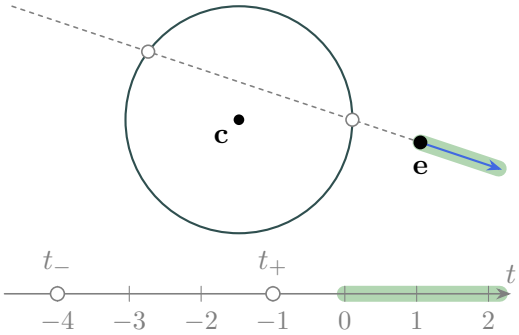

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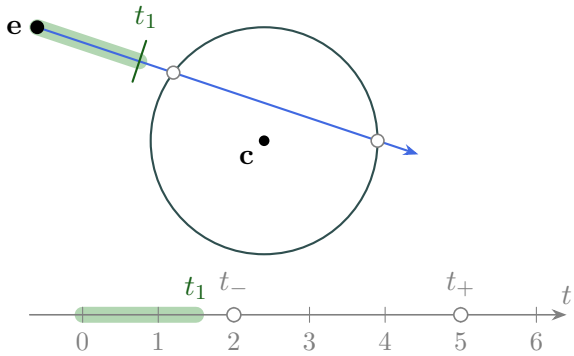
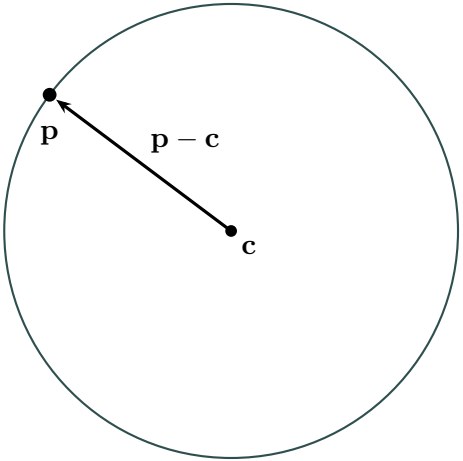


$[t_0, t_1] = [0, 1.5]$: $t = 2,\ 5$

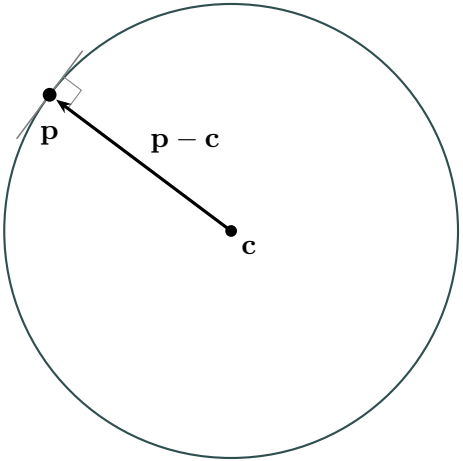
- Sphere too far: both roots above t_1

The normal points straight out from $\mathbf{c}$



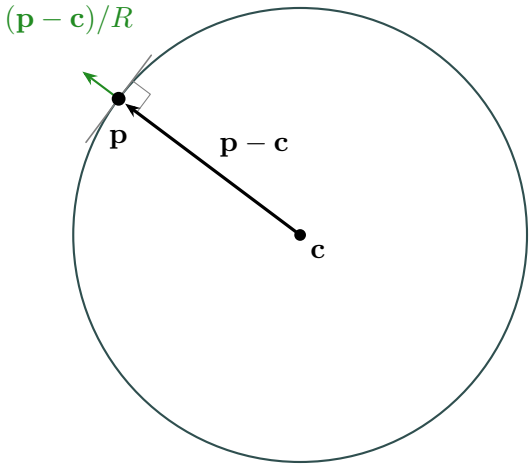
- $\mathbf{p} - \mathbf{c}$ points straight out at $\mathbf{p}$

The normal points straight out from $\mathbf{c}$



- $\mathbf{p} - \mathbf{c}$ points straight out at $\mathbf{p}$

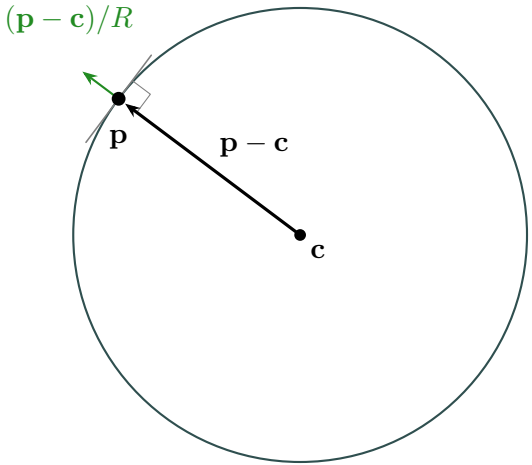
The normal points straight out from $\mathbf{c}$



- $\mathbf{p}-\mathbf{c}$ points straight out at $\mathbf{p}$
- Its length is R: divide by R

$$\text{unit normal} = \frac{\mathbf{p}-\mathbf{c}}{R}$$

The normal points straight out from $\mathbf{c}$

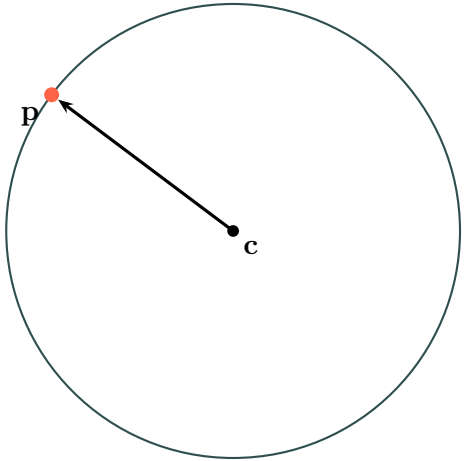


- $\mathbf{p}-\mathbf{c}$ points straight out at $\mathbf{p}$
- Its length is R: divide by R

$$\text{unit normal} = \frac{\mathbf{p}-\mathbf{c}}{R}$$

- Book: $\mathbf{n} = 2(\mathbf{p}-\mathbf{c})$, same direction

Worked: the unit normal where the ray enters

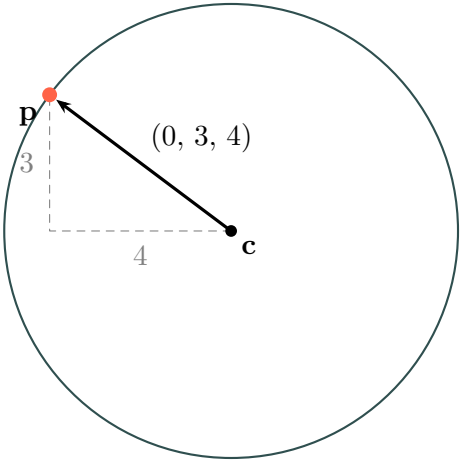


- Try first: $\mathbf{p} - \mathbf{c}$, length, unit normal

$$\mathbf{p} = (0,\ 3,\ -6), \quad \mathbf{c} = (0,\ 0,\ -10), \quad R = 5$$

- At the entry point, $t = 2$

Worked: the unit normal where the ray enters



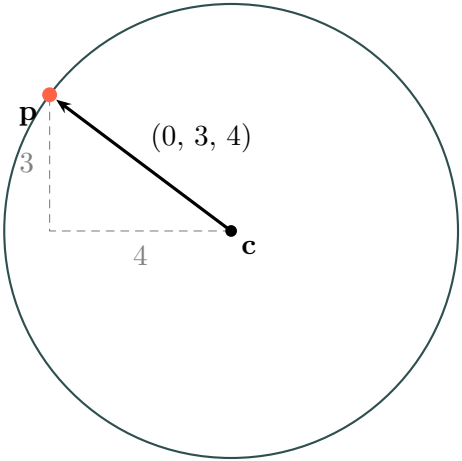
- Try first: $\mathbf{p} - \mathbf{c}$, length, unit normal

$$\mathbf{p} = (0,\ 3,\ -6), \quad \mathbf{c} = (0,\ 0,\ -10), \quad R = 5$$

$$\mathbf{p} - \mathbf{c} = (0, 3, -6) - (0, 0, -10) = (0,\ 3,\ 4)$$

- At the entry point, $t = 2$

Worked: the unit normal where the ray enters



- Try first: $\mathbf{p} - \mathbf{c}$, length, unit normal

$$\mathbf{p} = (0,\ 3,\ -6), \quad \mathbf{c} = (0,\ 0,\ -10), \quad R = 5$$

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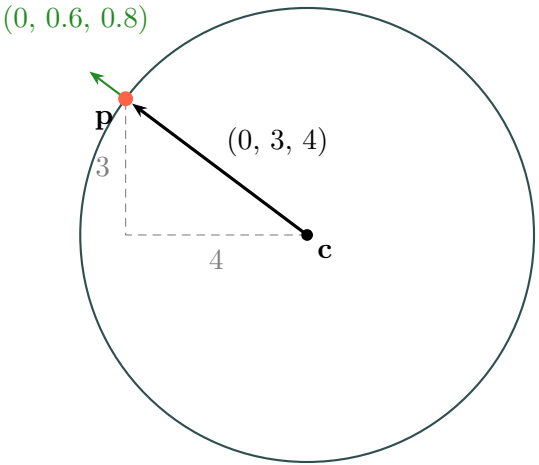
$$\|\mathbf{p} - \mathbf{c}\| = \sqrt{0 + 9 + 16} = \sqrt{25} = 5 = R$$

- At the entry point, $t = 2$

Worked: the unit normal where the ray enters



- Try first: $\mathbf{p} - \mathbf{c}$, length, unit normal

$$\mathbf{p} = (0,\ 3,\ -6), \quad \mathbf{c} = (0,\ 0,\ -10), \quad R = 5$$

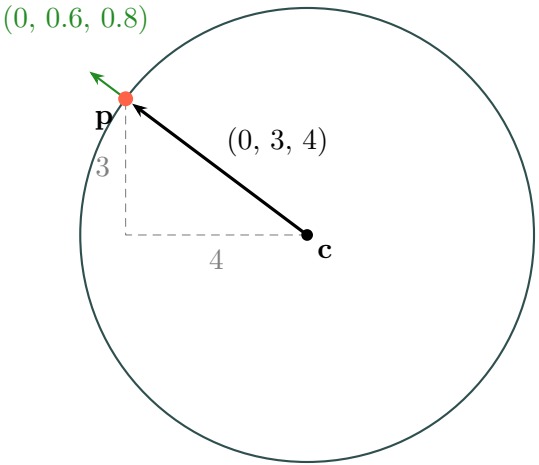
$$\mathbf{p} - \mathbf{c} = (0, 3, -6) - (0, 0, -10) = (0,\ 3,\ 4)$$

$$\|\mathbf{p} - \mathbf{c}\| = \sqrt{0 + 9 + 16} = \sqrt{25} = 5 = R$$

$$\frac{\mathbf{p} - \mathbf{c}}{R} = \frac{(0, 3, 4)}{5} = (0,\ 0.6,\ 0.8)$$

- At the entry point, $t = 2$
- Length 1: up 0.6, toward us 0.8

Worked: the unit normal where the ray enters



- Try first: $\mathbf{p} - \mathbf{c}$, length, unit normal

$$\mathbf{p} = (0,\ 3,\ -6), \quad \mathbf{c} = (0,\ 0,\ -10), \quad R = 5$$

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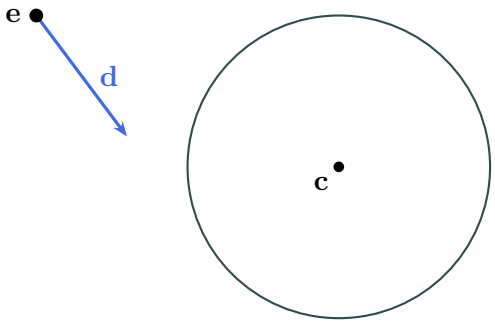
$$\|\mathbf{p} - \mathbf{c}\| = \sqrt{0 + 9 + 16} = \sqrt{25} = 5 = R$$

$$\frac{\mathbf{p} - \mathbf{c}}{R} = \frac{(0, 3, 4)}{5} = (0,\ 0.6,\ 0.8)$$

$$\mathbf{n} = 2\,(\mathbf{p} - \mathbf{c}) = (0,\ 6,\ 8)$$

- At the entry point, $t = 2$
- Length 1: up 0.6, toward us 0.8

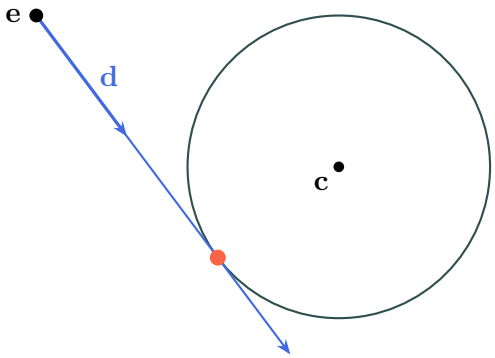
Quick check 2: hit, graze, or miss?



$$\mathbf{e} = (0, 5, 0), \quad \mathbf{d} = (0, -4, -3)$$
$$\mathbf{c} = (0, 0, -10), \quad R = 5$$

- Find A, B, C, then D
- If it touches: which t, where?

Quick check 2: hit, graze, or miss?



$$\mathbf{e} = (0, 5, 0), \quad \mathbf{d} = (0, -4, -3)$$
$$\mathbf{c} = (0, 0, -10), \quad R = 5$$

- Find A, B, C, then D
- If it touches: which t, where?

$A = 0 + 16 + 9 = 25$

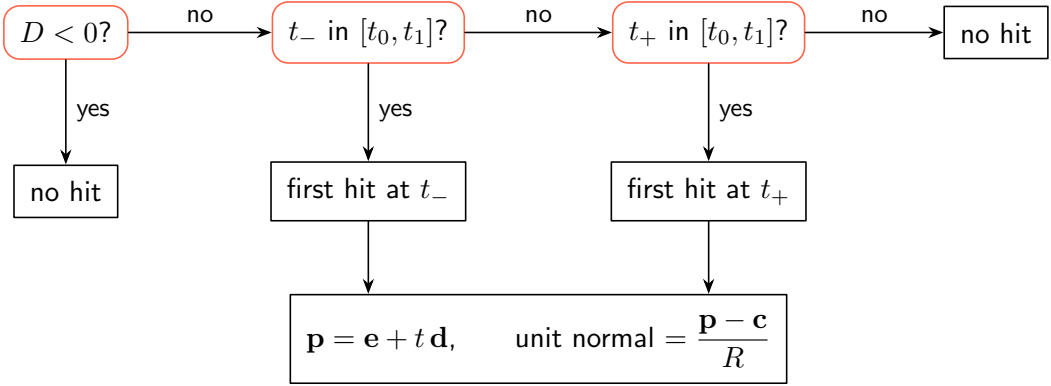
$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = 0 - 20 - 30 = -50, \quad B = -100$

$C = 100$ (as before)

$D = 10000 - 10000 = 0$: a graze

$$t = \frac{100}{50} = 2, \quad \mathbf{p} = \mathbf{e} + 2\,\mathbf{d} = (0, -3, -6)$$

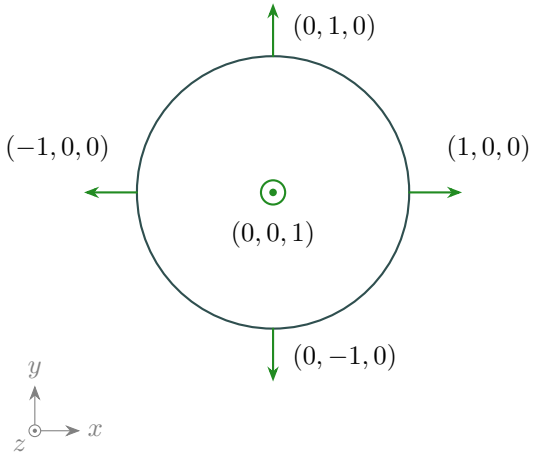
Recap: from two roots to the first hit



Colour by the normal

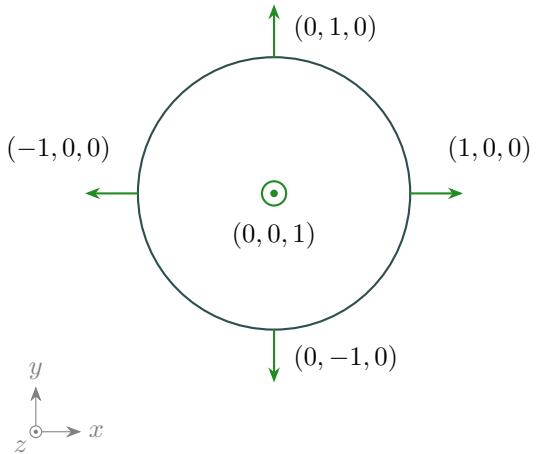
Our sphere's first shaded picture

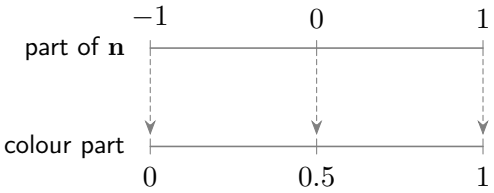
Show the normal as a colour



- No lights yet: colour by the normal

Show the normal as a colour

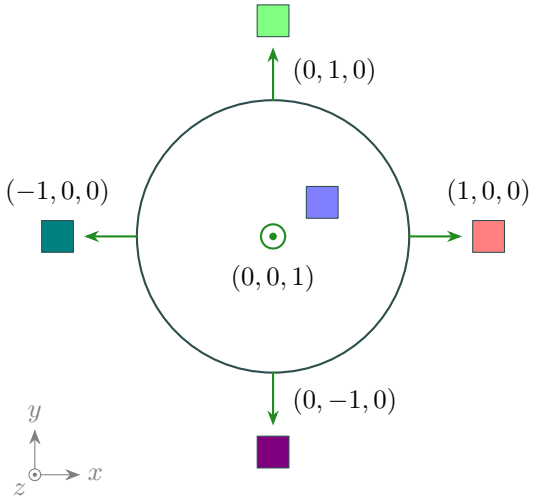


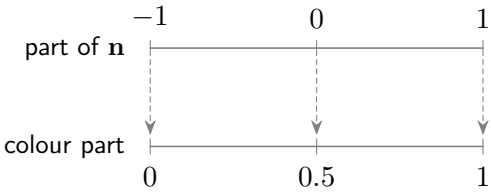


$$\text{colour} = \tfrac{1}{2}\left(\mathbf{n} + (1,\,1,\,1)\right)$$

- No lights yet: colour by the normal
- Each part: $[-1, 1]$ onto $[0, 1]$

Show the normal as a colour

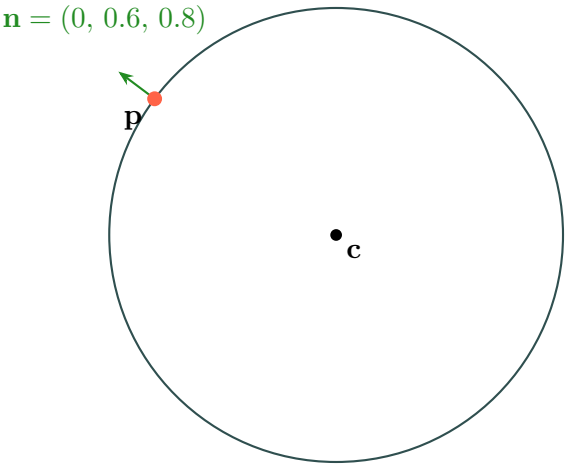




$$\text{colour} = \tfrac{1}{2}\left(\mathbf{n} + (1,\,1,\,1)\right)$$

- No lights yet: colour by the normal
- Each part: $[-1, 1]$ onto $[0, 1]$
- x red, y green, z blue

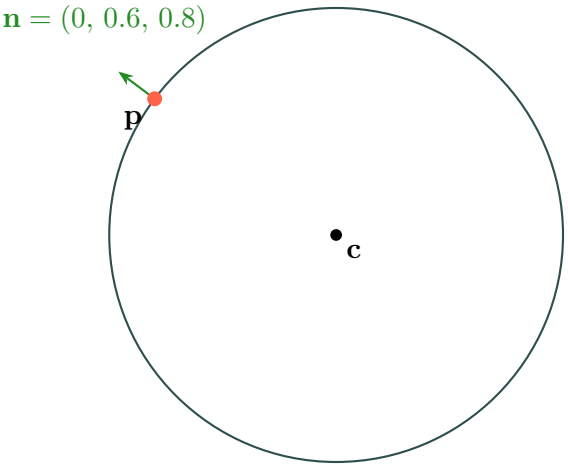
Worked: the colour where our ray enters



- Try first: the colour at $\mathbf{p}$

  colour $= \frac{1}{2}\big(\mathbf{n} + (1,\ 1,\ 1)\big)$

Worked: the colour where our ray enters



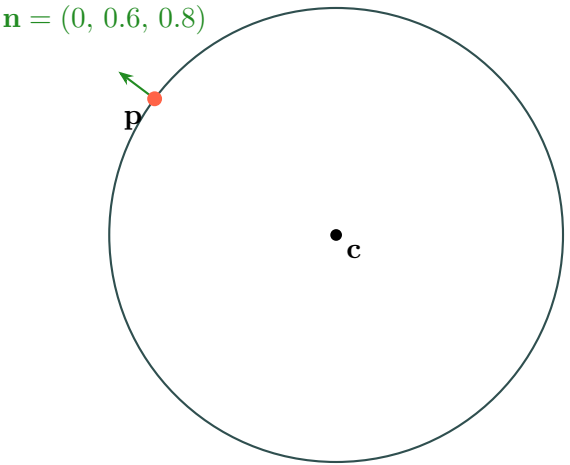
- Try first: the colour at $\mathbf{p}$

$$\text{colour} = \tfrac{1}{2}\big(\mathbf{n} + (1,\ 1,\ 1)\big)$$

$$\Big(\frac{0+1}{2},\ \frac{0.6+1}{2},\ \frac{0.8+1}{2}\Big)$$

$$= (0.5,\ 0.8,\ 0.9)$$

Worked: the colour where our ray enters



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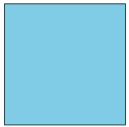
$$\text{colour} = \tfrac{1}{2}\big(\mathbf{n} + (1,\ 1,\ 1)\big)$$

$$\Big(\frac{0+1}{2},\ \frac{0.6+1}{2},\ \frac{0.8+1}{2}\Big)$$

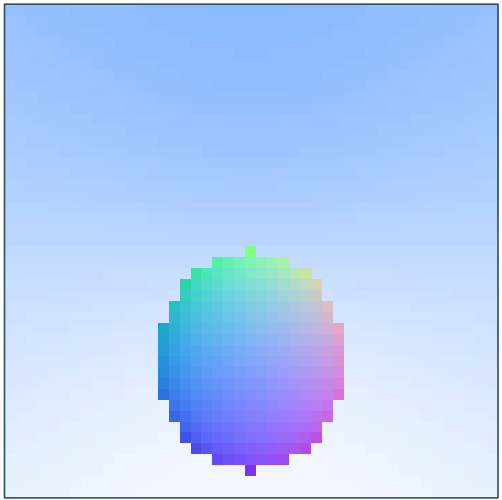
$$= (0.5,\ 0.8,\ 0.9)$$



red 0.5
green 0.8
blue 0.9

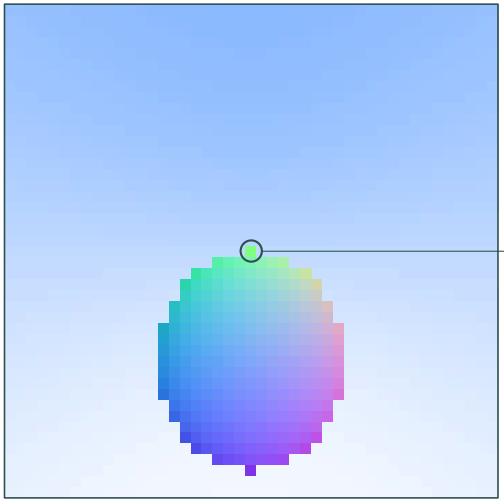
- Up and toward us: green and blue

Our sphere, coloured by its normals



- Every hit coloured by its normal

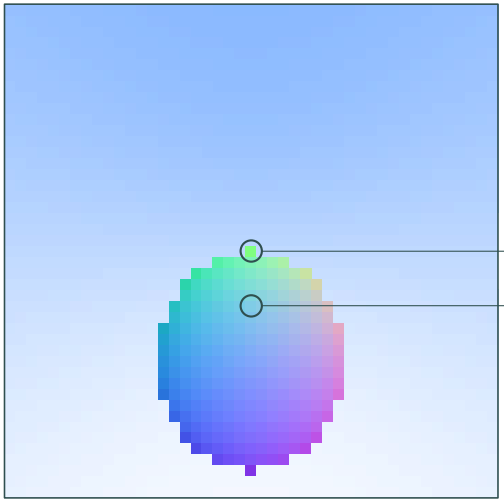
Our sphere, coloured by its normals



- Every hit coloured by its normal

top, faces up: $(0.5,\ 1,\ 0.5)$

Our sphere, coloured by its normals

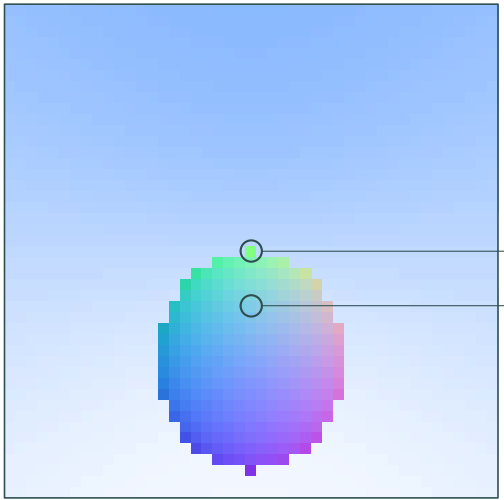


- Every hit coloured by its normal

top, faces up: $(0.5,\ 1,\ 0.5)$

our ray: $(0.5,\ 0.8,\ 0.9)$

Our sphere, coloured by its normals



- Every hit coloured by its normal
- Faces left: teal; faces right: pink

top, faces up: $(0.5,\ 1,\ 0.5)$

our ray: $(0.5,\ 0.8,\ 0.9)$

The sphere test, step by step

1 $A = \mathbf{d} \cdot \mathbf{d}, \quad B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}), \quad C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$ part one

The sphere test, step by step

1. $A = \mathbf{d} \cdot \mathbf{d}, \quad B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}), \quad C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$ — part one
2. $D = B^2 - 4AC$; if $D < 0$: no hit — sign of D

- A miss stops early: no square root

The sphere test, step by step

1. $A = \mathbf{d} \cdot \mathbf{d}, \quad B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}), \quad C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$ — part one
2. $D = B^2 - 4AC$; if $D < 0$: no hit — sign of D
3. $t = (-B - \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit at t — case 1

- A miss stops early: no square root

The sphere test, step by step

1. $A = \mathbf{d} \cdot \mathbf{d}, \quad B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}), \quad C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$ — part one
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4. else $t = (-B + \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit at t — case 2

- A miss stops early: no square root

The sphere test, step by step

1. $A = \mathbf{d} \cdot \mathbf{d}, \quad B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}), \quad C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$ — part one
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3. $t = (-B - \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit at t — case 1
4. else $t = (-B + \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit at t — case 2
5. else: no hit — case 3

- A miss stops early: no square root

The sphere test, step by step

1. $A = \mathbf{d} \cdot \mathbf{d}, \quad B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}), \quad C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$ — part one
2. $D = B^2 - 4AC$; if $D < 0$: no hit — sign of D
3. $t = (-B - \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit at t — case 1
4. else $t = (-B + \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit at t — case 2
5. else: no hit — case 3
6. at a hit: $\mathbf{p} = \mathbf{e} + t\,\mathbf{d}, \quad \mathbf{n} = (\mathbf{p} - \mathbf{c})/R$ — point and normal

- A miss stops early: no square root

The sphere test, step by step

1. $A = \mathbf{d} \cdot \mathbf{d}$, $B = 2\,\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$, $C = (\mathbf{e} - \mathbf{c}) \cdot (\mathbf{e} - \mathbf{c}) - R^2$ — part one
2. $D = B^2 - 4AC$; if $D < 0$: no hit — sign of D
3. $t = (-B - \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit at t — case 1
4. else $t = (-B + \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit at t — case 2
5. else: no hit — case 3
6. at a hit: $\mathbf{p} = \mathbf{e} + t\,\mathbf{d}$, $\mathbf{n} = (\mathbf{p} - \mathbf{c})/R$ — point and normal
7. colour $= \frac{1}{2}\big(\mathbf{n} + (1, 1, 1)\big)$; no hit: the sky — part three

- A miss stops early: no square root
- A ray tracer for one sphere

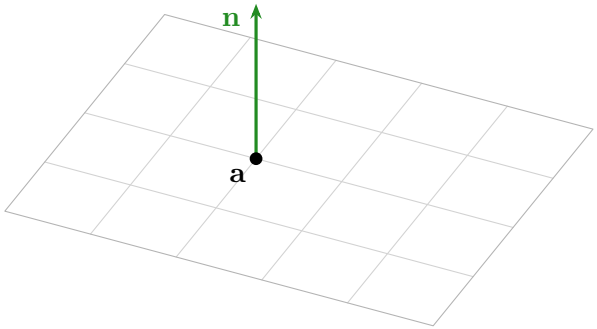
Recap: from hit to colour

$$\mathbf{p} = \mathbf{e} + t\,\mathbf{d} \;\longrightarrow\; \mathbf{n} = \frac{\mathbf{p} - \mathbf{c}}{R} \;\longrightarrow\; \text{colour} = \frac{\mathbf{n} + (1,1,1)}{2}$$

The ray meets a plane

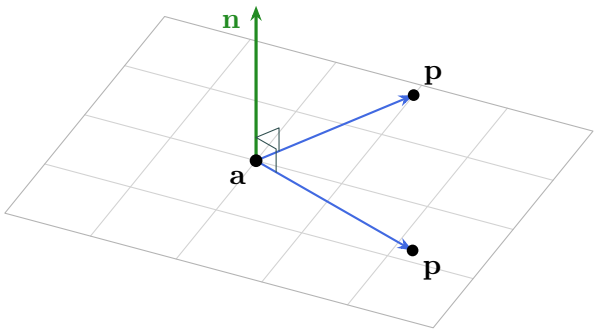
The same move, with one power of t

A plane: arrows in it are perpendicular to $\mathbf{n}$



- Point $\mathbf{a}$ on it, normal $\mathbf{n}$

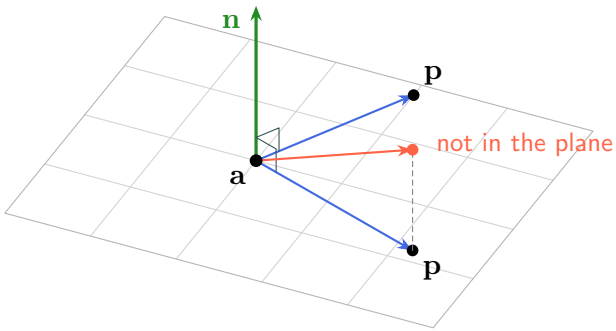
A plane: arrows in it are perpendicular to $\mathbf{n}$



$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = 0$$

- Point $\mathbf{a}$ on it, normal $\mathbf{n}$
- In it: $\mathbf{p} - \mathbf{a}$ perpendicular to $\mathbf{n}$

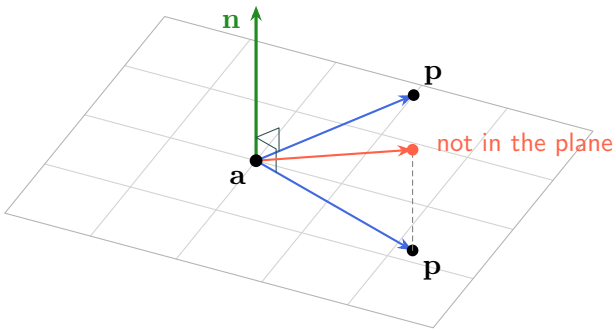
A plane: arrows in it are perpendicular to $\mathbf{n}$



$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = 0$$

- Point $\mathbf{a}$ on it, normal $\mathbf{n}$
- In it: $\mathbf{p} - \mathbf{a}$ perpendicular to $\mathbf{n}$
- Off it: no right angle

A plane: arrows in it are perpendicular to $\mathbf{n}$



$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = 0$$

- Point $\mathbf{a}$ on it, normal $\mathbf{n}$
- In it: $\mathbf{p} - \mathbf{a}$ perpendicular to $\mathbf{n}$
- Off it: no right angle

our floor, under the sphere

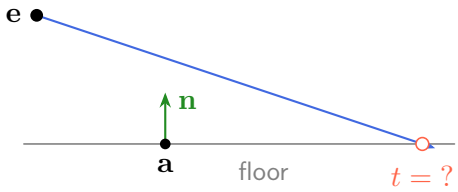
$$\mathbf{a} = (0, -5, -10), \quad \mathbf{n} = (0, 1, 0)$$

$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = y + 5 = 0$$

every point with $y = -5$

Put the ray in: only one power of t

$$(\mathbf{p}(t) - \mathbf{a}) \cdot \mathbf{n} = 0$$

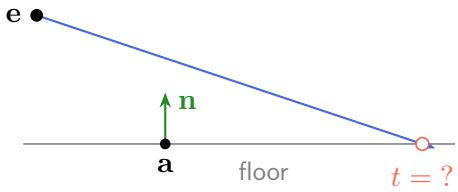


- A hit: $\mathbf{p}(t)$ fits the plane

Put the ray in: only one power of t

$$(\mathbf{p}(t) - \mathbf{a}) \cdot \mathbf{n} = 0$$

$$(\mathbf{e} + t\,\mathbf{d} - \mathbf{a}) \cdot \mathbf{n} = 0 \qquad \text{put in } \mathbf{p}(t)$$



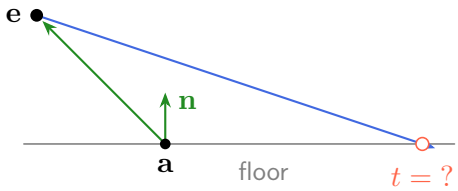
- A hit: $\mathbf{p}(t)$ fits the plane

Put the ray in: only one power of t

$$(\mathbf{p}(t) - \mathbf{a}) \cdot \mathbf{n} = 0$$

$$(\mathbf{e} + t\,\mathbf{d} - \mathbf{a}) \cdot \mathbf{n} = 0 \qquad \text{put in } \mathbf{p}(t)$$

$$\big((\mathbf{e} - \mathbf{a}) + t\,\mathbf{d}\big) \cdot \mathbf{n} = 0 \qquad \text{group, as before}$$



- A hit: $\mathbf{p}(t)$ fits the plane
- Same move as the sphere

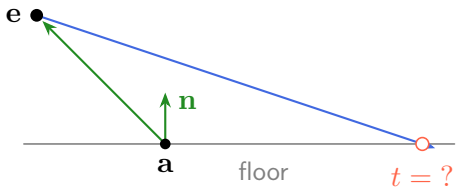
Put the ray in: only one power of t

$$(\mathbf{p}(t) - \mathbf{a}) \cdot \mathbf{n} = 0$$

$$(\mathbf{e} + t\,\mathbf{d} - \mathbf{a}) \cdot \mathbf{n} = 0 \qquad \text{put in } \mathbf{p}(t)$$

$$\big((\mathbf{e} - \mathbf{a}) + t\,\mathbf{d}\big) \cdot \mathbf{n} = 0 \qquad \text{group, as before}$$

$$(\mathbf{e} - \mathbf{a}) \cdot \mathbf{n} + t\,(\mathbf{d} \cdot \mathbf{n}) = 0 \qquad \text{multiply out}$$



- A hit: $\mathbf{p}(t)$ fits the plane
- Same move as the sphere

Put the ray in: only one power of t

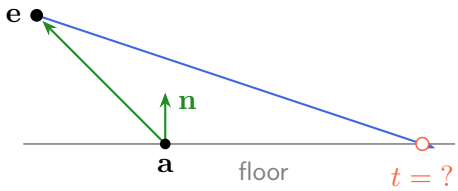
$$(\mathbf{p}(t) - \mathbf{a}) \cdot \mathbf{n} = 0$$

$$(\mathbf{e} + t\,\mathbf{d} - \mathbf{a}) \cdot \mathbf{n} = 0 \qquad \text{put in } \mathbf{p}(t)$$

$$\big((\mathbf{e} - \mathbf{a}) + t\,\mathbf{d}\big) \cdot \mathbf{n} = 0 \qquad \text{group, as before}$$

$$(\mathbf{e} - \mathbf{a}) \cdot \mathbf{n} + t\,(\mathbf{d} \cdot \mathbf{n}) = 0 \qquad \text{multiply out}$$

$$t\,(\mathbf{d} \cdot \mathbf{n}) = (\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} \qquad \text{move over}$$



- A hit: $\mathbf{p}(t)$ fits the plane
- Same move as the sphere

Put the ray in: only one power of t

$$(\mathbf{p}(t) - \mathbf{a}) \cdot \mathbf{n} = 0$$

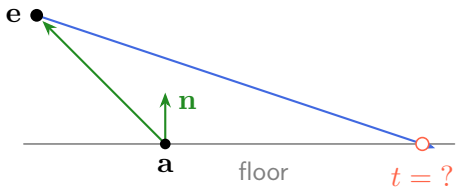
$$(\mathbf{e} + t\,\mathbf{d} - \mathbf{a}) \cdot \mathbf{n} = 0 \qquad \text{put in } \mathbf{p}(t)$$

$$\big((\mathbf{e} - \mathbf{a}) + t\,\mathbf{d}\big) \cdot \mathbf{n} = 0 \qquad \text{group, as before}$$

$$(\mathbf{e} - \mathbf{a}) \cdot \mathbf{n} + t\,(\mathbf{d} \cdot \mathbf{n}) = 0 \qquad \text{multiply out}$$

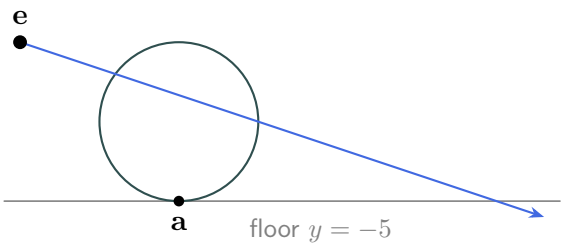
$$t\,(\mathbf{d} \cdot \mathbf{n}) = (\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} \qquad \text{move over}$$

$$t = \frac{(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n}}{\mathbf{d} \cdot \mathbf{n}} \qquad \text{divide by } \mathbf{d} \cdot \mathbf{n}$$



- A hit: $\mathbf{p}(t)$ fits the plane
- Same move as the sphere
- One power of t: one crossing

Worked: where our ray meets the floor

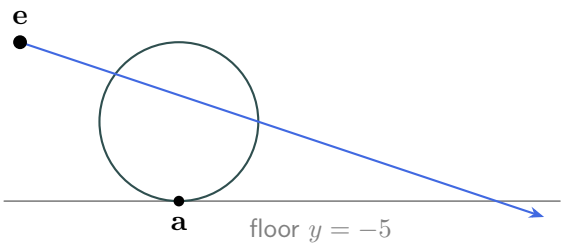


- Try first: $(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n}$, $\mathbf{d} \cdot \mathbf{n}$, t, the point

$\mathbf{e} = (0, 5, 0), \quad \mathbf{d} = (0, -1, -3)$

$\mathbf{a} = (0, -5, -10), \quad \mathbf{n} = (0, 1, 0)$

Worked: where our ray meets the floor



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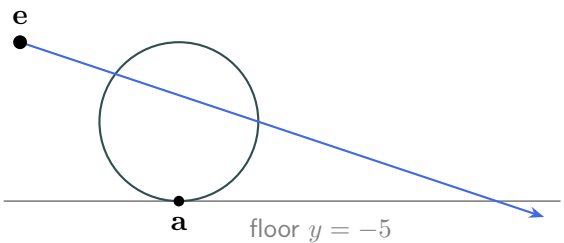
$$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = (0, -10, -10) \cdot (0, 1, 0) = -10$$

Worked: where our ray meets the floor



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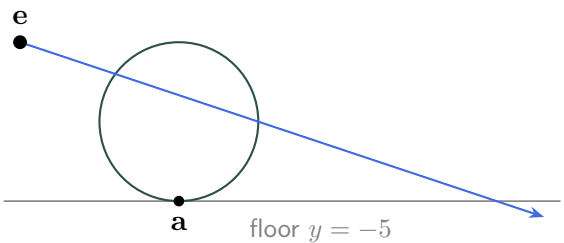
$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = (0, -10, -10) \cdot (0, 1, 0) = -10$

$\mathbf{d} \cdot \mathbf{n} = (0, -1, -3) \cdot (0, 1, 0) = -1$

Worked: where our ray meets the floor



- Try first: $(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n}$, $\mathbf{d} \cdot \mathbf{n}$, t, the point

$\mathbf{e} = (0, 5, 0), \quad \mathbf{d} = (0, -1, -3)$

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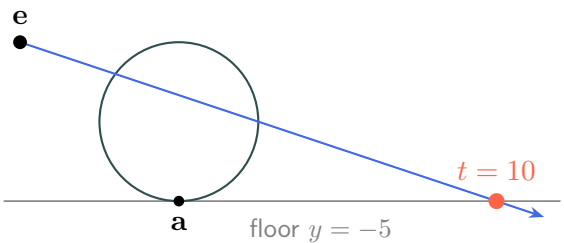
$$t = \frac{-10}{-1} = 10$$

Worked: where our ray meets the floor



- On the floor: $y = -5$

- Try first: $(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n}$, $\mathbf{d} \cdot \mathbf{n}$, t, the point

$$\mathbf{e} = (0, 5, 0), \quad \mathbf{d} = (0, -1, -3)$$
$$\mathbf{a} = (0, -5, -10), \quad \mathbf{n} = (0, 1, 0)$$

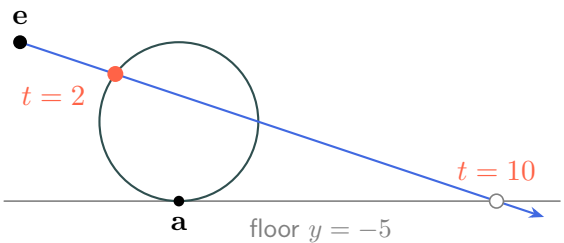
$$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = (0, -10, -10) \cdot (0, 1, 0) = -10$$

$$\mathbf{d} \cdot \mathbf{n} = (0, -1, -3) \cdot (0, 1, 0) = -1$$

$$t = \frac{-10}{-1} = 10$$

$$\mathbf{p}(10) = (0, 5, 0) + 10\,(0, -1, -3)$$
$$= (0, -5, -30)$$

Worked: where our ray meets the floor



- On the floor: $y = -5$
- But the sphere comes first, at $t = 2$

- Try first: $(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n}$, $\mathbf{d} \cdot \mathbf{n}$, t, the point

$$\mathbf{e} = (0, 5, 0), \quad \mathbf{d} = (0, -1, -3)$$
$$\mathbf{a} = (0, -5, -10), \quad \mathbf{n} = (0, 1, 0)$$

$$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = (0, -10, -10) \cdot (0, 1, 0) = -10$$

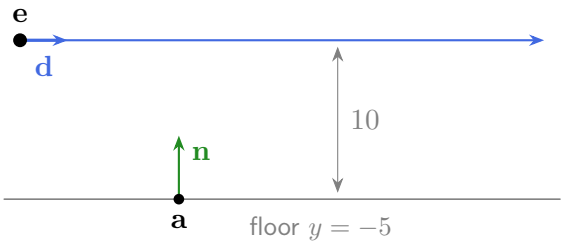
$$\mathbf{d} \cdot \mathbf{n} = (0, -1, -3) \cdot (0, 1, 0) = -1$$

$$t = \frac{-10}{-1} = 10$$

$$\mathbf{p}(10) = (0, 5, 0) + 10\,(0, -1, -3)$$
$$= (0, -5, -30)$$

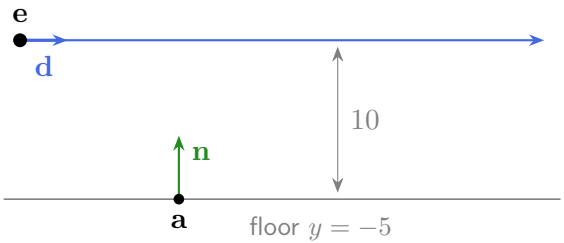
A parallel ray never meets the plane

$\mathbf{d} = (0,\ 0,\ -3)$ the middle pixel's ray



- Level ray: stays at height 5

A parallel ray never meets the plane

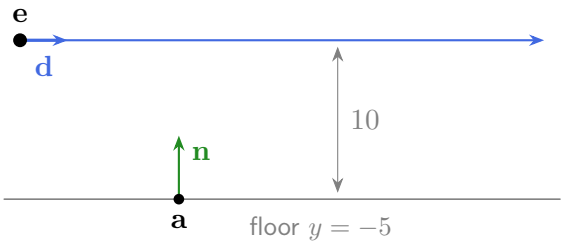


$\mathbf{d} = (0,\ 0,\ -3)$ the middle pixel's ray

$\mathbf{d} \cdot \mathbf{n} = (0, 0, -3) \cdot (0, 1, 0) = 0$

- Level ray: stays at height 5
- $\mathbf{d} \cdot \mathbf{n} = 0$: runs along the plane

A parallel ray never meets the plane



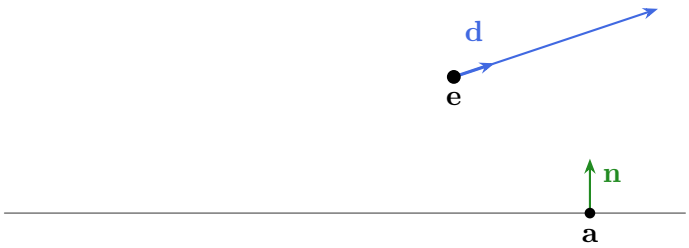
$\mathbf{d} = (0,\ 0,\ -3)$ the middle pixel's ray

$\mathbf{d} \cdot \mathbf{n} = (0, 0, -3) \cdot (0, 1, 0) = 0$

$t = \dfrac{-10}{0}$: no such number

- Level ray: stays at height 5
- $\mathbf{d} \cdot \mathbf{n} = 0$: runs along the plane
- Check $\mathbf{d} \cdot \mathbf{n}$ first: zero, no hit

Worked: the top ray and the floor

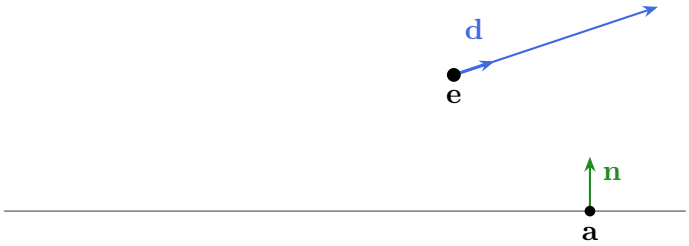


- Try first: $\mathbf{d} \cdot \mathbf{n}$, t, where?

$\mathbf{d} = (0,\ 1,\ -3)$

$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = -10$

Worked: the top ray and the floor



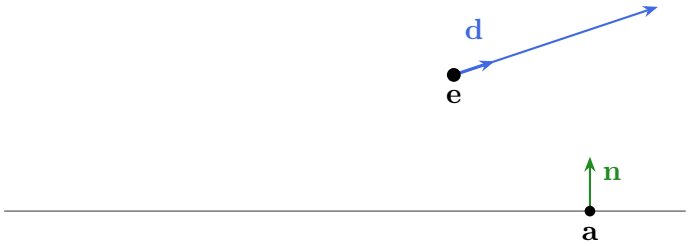
- Try first: $\mathbf{d} \cdot \mathbf{n}$, t, where?

$\mathbf{d} = (0,\ 1,\ -3)$

$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = -10$

$\mathbf{d} \cdot \mathbf{n} = 1$

Worked: the top ray and the floor



- Try first: $\mathbf{d} \cdot \mathbf{n}$, t, where?

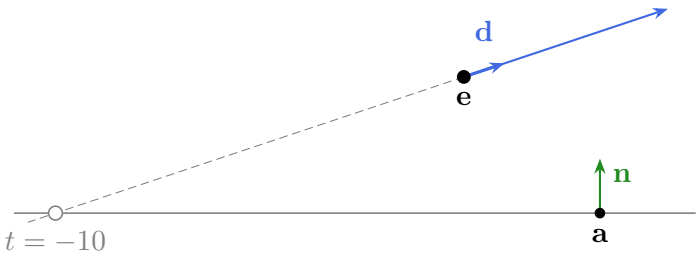
$\mathbf{d} = (0,\ 1,\ -3)$

$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = -10$

$\mathbf{d} \cdot \mathbf{n} = 1$

$$t = \frac{-10}{1} = -10$$

Worked: the top ray and the floor



- The crossing lies behind the eye

- Try first: $\mathbf{d} \cdot \mathbf{n}$, t, where?

$$\mathbf{d} = (0,\ 1,\ -3)$$

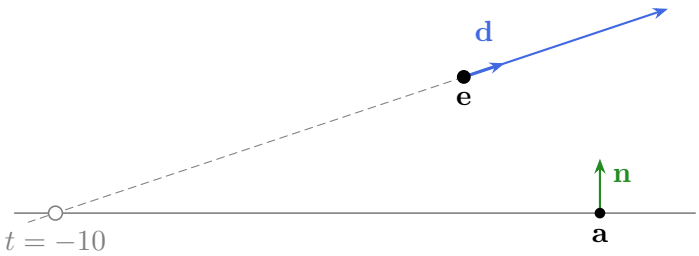
$$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = -10$$

$$\mathbf{d} \cdot \mathbf{n} = 1$$

$$t = \frac{-10}{1} = -10$$

$$\mathbf{p}(-10) = (0,\ -5,\ 30)$$

Worked: the top ray and the floor



- The crossing lies behind the eye
- $t < t_0 = 0$: not a hit

- Try first: $\mathbf{d} \cdot \mathbf{n}$, t, where?

$\mathbf{d} = (0,\ 1,\ -3)$

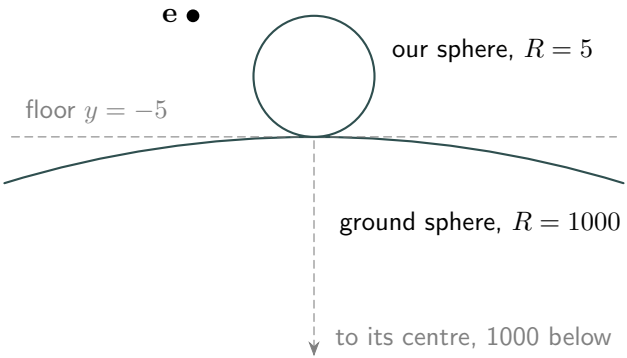
$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n} = -10$

$\mathbf{d} \cdot \mathbf{n} = 1$

$$t = \frac{-10}{1} = -10$$

$\mathbf{p}(-10) = (0,\ -5,\ 30)$

A huge sphere can stand in for a floor



not to scale

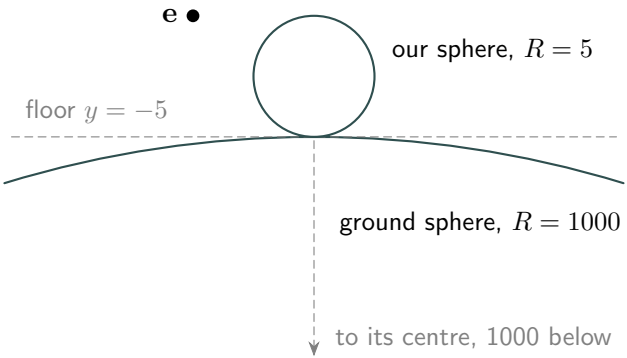
ground: $\mathbf{c} = (0, -1005, -10)$

$R = 1000$, top at $(0, -5, -10)$

Ray Tracing in One Weekend:
ball 0.5, ground 100

- Seen from close by: almost flat

A huge sphere can stand in for a floor



not to scale

ground: $\mathbf{c} = (0, -1005, -10)$

$R = 1000$, top at $(0, -5, -10)$

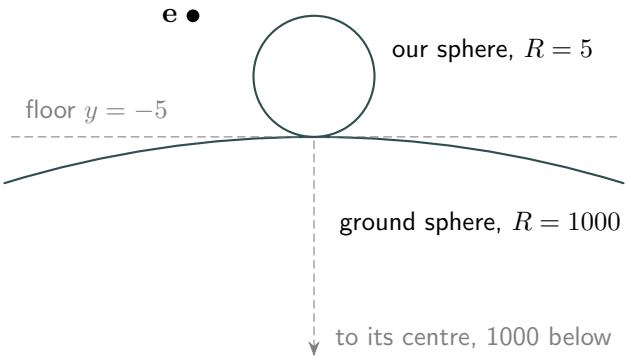
Ray Tracing in One Weekend:
ball 0.5, ground 100

our ray: plane $t = 10$

ground sphere $t \approx 10.2$

- Seen from close by: almost flat

A huge sphere can stand in for a floor



not to scale

ground: $\mathbf{c} = (0, -1005, -10)$

$R = 1000$, top at $(0, -5, -10)$

Ray Tracing in One Weekend:
ball 0.5, ground 100

our ray: plane $t = 10$

ground sphere $t \approx 10.2$

- Seen from close by: almost flat
- Only the sphere test needed

Recap: the ray meets a plane

$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = 0 \quad \Longrightarrow \quad t = \frac{(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n}}{\mathbf{d} \cdot \mathbf{n}}$$

Summary

1 Ray into sphere

$$A\,t^2 + B\,t + C = 0$$

$$A = \mathbf{d}\cdot\mathbf{d}, \quad B = 2\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$$

$$C = (\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c}) - R^2$$

2 Solve; the sign of D

$$t = \frac{-B \pm \sqrt{D}}{2A}$$

$$D = B^2 - 4AC$$

< 0 miss, $= 0$ graze, > 0 hit

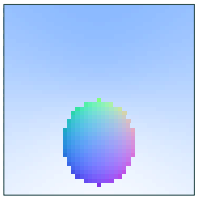
3 First hit in $[t_0, t_1]$

t_- if inside, else t_+,

else no hit

behind the eye: $t < 0$, no hit

4 One ray per pixel



5 Normal, then colour

$$\mathbf{n} = \frac{\mathbf{p}-\mathbf{c}}{R}$$

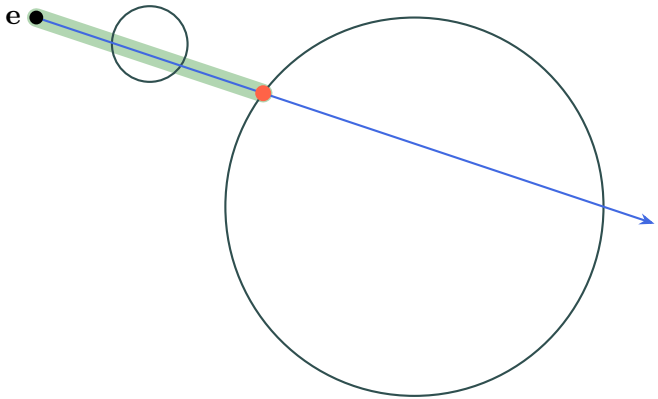
$$\text{colour} = \frac{\mathbf{n} + (1,1,1)}{2}$$

6 Ray into plane

$$t = \frac{(\mathbf{a}-\mathbf{e})\cdot\mathbf{n}}{\mathbf{d}\cdot\mathbf{n}}$$

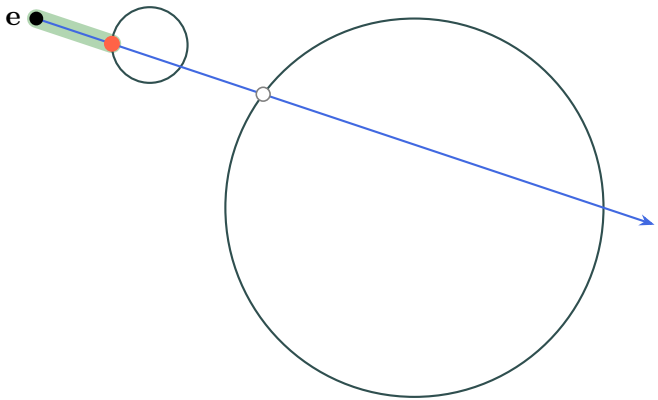
$\mathbf{d}\cdot\mathbf{n} = 0$: parallel, no hit

Next lecture: many objects, one nearest hit



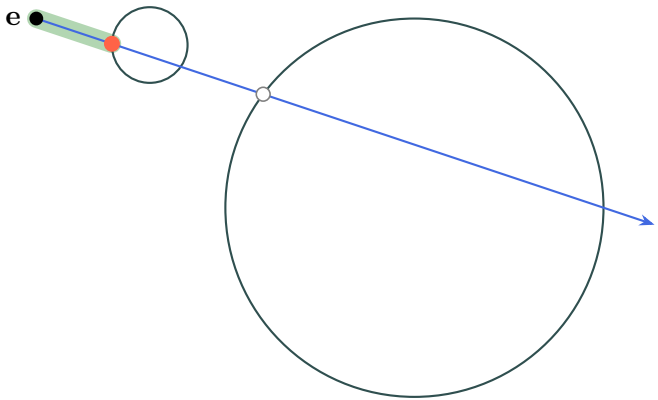
- Test the ray against every object

Next lecture: many objects, one nearest hit



- Test the ray against every object
- Keep the nearest: t_1 shrinks

Next lecture: many objects, one nearest hit



- Test the ray against every object
- Keep the nearest: t_1 shrinks
- Then: smooth edges, several rays per pixel