Finger Exercises 14

Ray-Sphere Intersection, and a Floor

≈ 60 minutes, plus a bonus
CS 453 — Computer Graphics

You need a pencil, a ruler and some coloured pencils. Every number was chosen to come out by hand; only the bonus is easier with a calculator. The letters are the lecture's: the ray $\mathbf{p}(t) = \mathbf{e} + t\,\mathbf{d}$ from the eye $\mathbf{e}$, a sphere with centre $\mathbf{c}$ and radius R, and

$$A = \mathbf{d}\cdot\mathbf{d}, \qquad B = 2\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c}), \qquad C = (\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c}) - R^2, \qquad D = B^2 - 4AC, \qquad t_\pm = \frac{-B \pm \sqrt{D}}{2A}.$$

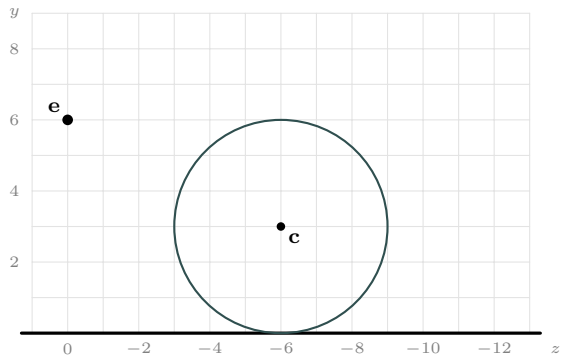
Nothing here is an assignment task; the sheet is Lecture 14 only.

Short on time? Do Exercises 3, 4, 7 and 9 (about 30 minutes): five rays against one ball, which root counts, normals as colours, and one ray against five planes.

Part 1 · The ray meets the sphere

Exercises 1–3

Ball B is used on the whole sheet. It has radius $R = 3$ and rests on the floor $y = 0$, so its centre is $\mathbf{c} = (0, 3, -6)$ and it touches the floor at $(0, 0, -6)$. The eye sits at $\mathbf{e} = (0, 6, 0)$, 6 above the floor, and looks along $-z$.
The side view shows every point with $x = 0$ at its true size: $-z$ runs to the right, y up, and x points toward you. The numbers under the floor are z values.



1 Inside, on, or outside?

about 3 min

The sphere as one dot product: its sign tells inside from outside before any ray is involved.

A point $\mathbf{p}$ is on Ball B when $(\mathbf{p}-\mathbf{c})\cdot(\mathbf{p}-\mathbf{c}) = R^2$. So the number $(\mathbf{p}-\mathbf{c})\cdot(\mathbf{p}-\mathbf{c}) - R^2$ is 0 on the ball, negative inside and positive outside.

point $\mathbf{p}$	$\mathbf{p}-\mathbf{c}$	$(\mathbf{p}-\mathbf{c})\cdot(\mathbf{p}-\mathbf{c}) - 9$	inside, on or outside?
$(0, 6, 0)$, the eye	$(0, 3, 6)$	$0 + 9 + 36 - 9 = 36$	outside
$(0, 3, -3)$			
$(0, 4, -5)$			
$(2, 1, -4)$			
$(-1, 5, -4)$			
$(0, 0, -6)$			
$(0, 3, -6)$			

(a) Which point gives the smallest number of all, and why can no point give less?

(b) The eye's number, 36, turns up again in Exercise 3. Under which of the letters A, B, C, D?

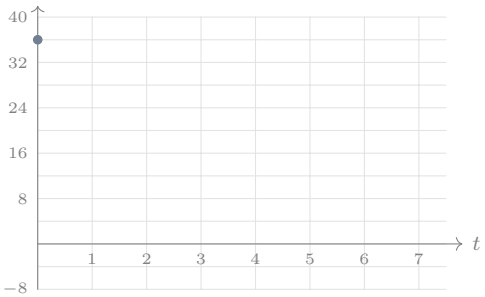
2 One ray, one parabola

about 6 min

The number from Exercise 1, taken along a ray, is the ray's quadratic: it crosses zero where the ray meets the ball.

Follow the ray from $\mathbf{e}$ with $\mathbf{d} = (0, -1, -1)$: one step down for every step into the scene.

t	0	1	2	3	4	5	6	7
$\mathbf{p}(t)$	$(0, 6, 0)$							
$\mathbf{p}(t) - \mathbf{c}$	$(0, 3, 6)$							
$(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) - 9$	36							



(a) Fill the table, then plot the eight numbers and join them with a smooth curve.

(b) Where does the curve cross zero? Mark those two points of the ray in the side view of Ball B. For which t is the ray inside the ball?

(c) The curve is $A\,t^2 + B\,t + C$. Read C off the plot at $t = 0$, then find A and B with the lecture's formulas. Check the curve's value at $t = 1$.

3 Five rays, one ball

about 10 min

Build the quadratic from dot products until it is automatic, and read miss, graze or hit from the sign of D.

Every ray starts at $\mathbf{e}$, so $\mathbf{e} - \mathbf{c} = (0, 3, 6)$ in every row. Compute D before any root: if it is negative, stop. The first row is done for you.

	$\mathbf{d}$	A	$\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$	B	C	D	miss, graze, hit?	t_-	t_+
	$(0, -1, -1)$	2	-9	-18	36	36	hit	3	6
1	$(0, 0, -1)$								
2	$(0, -2, -1)$								
3	$(1, -1, -2)$								
4	$(0, -4, -3)$								
5	$(2, -1, -5)$								

(a) Why is C the same in every row? Where did you meet its value before?

(b) Rows 1, 2 and 4, and the worked row, have $x = 0$: draw them in the side view. Mark where each one touches or enters the ball. The two grazing rays touch the ball at two points: which? What do these two rays mean for the ball's picture?

(c) Row 2 heads for the floor just in front of the ball. Where does it land? (Put $y = 0$ in $\mathbf{p}(t)$.)

(d) Row 5 is inside the ball only from t_- to t_+. What does that tell you about where its pixel sits in the ball's picture?

Part 2 · Solving for t

Exercises 4–6

4 Which root counts?

about 4 min

The book's rule for the first hit in an interval: smaller root, else larger root, else no hit.

Each row gives a ray's two roots and the interval $[t_0,\ t_1]$ it may use.

	t_-	t_+	$[t_0,\ t_1]$	first hit: t_-, t_+ or none?	the ray starts …
	3	6	$[0,\ +\infty)$	$t_- = 3$	outside, ball ahead
1	-1	2	$[0,\ +\infty)$		
2	-6	-3	$[0,\ +\infty)$		
3	3	6	$[0,\ 2.5]$		
4	3	6	$[4,\ +\infty)$		
5	1.2	1.2	$[0,\ +\infty)$		
6	0	3	$[0.001,\ +\infty)$		
7	none: $D < 0$		$[0,\ +\infty)$		
8	2	3	$[0,\ 2]$		

(a) In the last column, say where the ray starts: outside with the ball ahead, inside the ball, on its surface, or past it.

(b) Row 6 starts on the ball's surface, as a shadow ray from a point of the ball does. Why is its t_0 a small number and not 0?

(c) Rows 2, 3 and 7 all end with no hit. Which of them does D alone catch, and which need the interval?

5 What if . . . ?

about 6 min

Predict how A, B, C, D and the roots move when the ray or the ball changes, before computing anything.

Start from the worked ray of Exercise 3: $\mathbf{e} = (0, 6, 0)$, $\mathbf{d} = (0, -1, -1)$, with $A = 2$, $B = -18$, $C = 36$, $D = 36$ and roots 3 and 6. Change one thing at a time. Predict first, then check the row with the formulas.

	what changes	A	B	C	D	t_-	t_+	first hit
1	$\mathbf{d}$ doubled: $(0, -2, -2)$							
2	$\mathbf{d}$ reversed: $(0, 1, 1)$							
3	the eye slides 2 steps along its ray, to $(0, 4, -2)$							
4	the eye slides 3 steps, to $(0, 3, -3)$							
5	the eye slides 4 steps, to $(0, 2, -4)$							
6	the ball moves to $\mathbf{c} = (0, 9, 6)$, behind the eye							

(a) In rows 3 to 5, which numbers stay the same, and how do the roots move? Why must D stay the same while the eye slides along its own ray? (Hint: what does $t_+ - t_-$ measure?)

(b) Rows 2 and 6 give the same numbers. Why? Would a program that only tests $D \geq 0$ draw the ball of row 6?

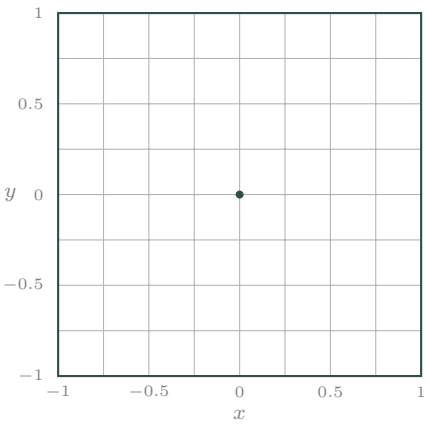
(c) Where is the eye in rows 4 and 5: outside, on or inside the ball? Which case of the book's rule does each row need?

6 A picture by hand

about 9 min

One ray per pixel, the sign of D decides the colour: the lecture's first picture, rendered with a pencil.

A simpler camera for this exercise: the eye sits at the origin and looks along $-z$. A ball of radius 3 has its centre at $(0,\ 0,\ -5)$, straight ahead. The window lies 1 unit ahead and runs from -1 to 1 in x and in y, with 8×8 pixels of side 0.25. The ray through the pixel whose centre is at (x, y) on the window has $\mathbf{d} = (x,\ y,\ -1)$.



(a) Show that $A = 1 + x^2 + y^2$, $B = -10$ and $C = 16$, so that $D = 36 - 64\,(x^2 + y^2)$.

(b) So $D \geq 0$ exactly when the pixel's centre lies inside a circle on the window. What are its centre and radius?

(c) Shade every pixel whose ray hits the ball. How many are there?

(d) Check the circle's radius with a right triangle: the eye, the ball's centre, and the point where a ray from the eye just grazes the ball. (Its sides are 3, 4 and 5.)

(e) Why is the edge of your picture a staircase? What would 16×16 pixels change, and what would they not change?

(f) The rays $(0,\ 3,\ -4)$ and $(0,\ -3,\ -4)$ miss the ball. Find each one's sky colour with the lecture's blend, $a = \frac{1}{2}(\hat{y} + 1)$ and $(1 - a)\,(1, 1, 1) + a\,(0.5,\ 0.7,\ 1)$. Which is paler, and why is the sky palest near the ground?

Part 3 · Colour by the normal

Exercises 7–8

7 Normals to colours, and back

about 7 min

The unit normal $(\mathbf{p} - \mathbf{c})/R$ and the lecture's colour map, run forwards and then backwards.

Unit normal $\mathbf{n} = (\mathbf{p} - \mathbf{c})/R$; colour $= \frac{1}{2}\big(\mathbf{n} + (1,\ 1,\ 1)\big)$. Backwards: $\mathbf{n} = 2 \times \text{colour} - (1,\ 1,\ 1)$.

Forwards. The entry points of Exercise 3's hits and grazes.

ray of Exercise 3	entry point $\mathbf{p}$	$\mathbf{p} - \mathbf{c}$	$\mathbf{n}$	colour
worked row, $t = 3$	$(0,\ 3,\ -3)$	$(0,\ 0,\ 3)$	$(0,\ 0,\ 1)$	$(0.5,\ 0.5,\ 1)$
row 1, $t = 6$				
row 3, $t = 2$				
row 4, $t = 1.2$				
row 5, $t = 1$				

Backwards: the colour detective. Six pixels from the middle of a normal picture, where every ray runs almost straight along $-z$. Two programs made them, one correct and one buggy.

	colour	**n**	length 1?	faces (up, left, toward us, ...)	in a correct picture?
A	(0.5, 0.9, 0.8)				
B	(0.2, 0.5, 0.9)				
C	(0.5, 0.5, 0.5)				
D	(0.5, 0.1, 0.2)				
E	(1, 1, 1)				
F	(0.9, 0.5, 0.8)				

(a) Which three colours can never appear in a correct picture? For each, guess what the buggy program did.

(b) Colour D has a unit normal, yet it is still wrong here. Why? (Think about which way a normal must face the eye.)

8 Symptoms and the broken step

about 5 min

Debugging from the picture: each wrong picture points to one step of the sphere test.

The sphere test, as in the lecture. **1** A, B, C from the ray and the ball. **2** $D = B^2 - 4AC$; if $D < 0$: no hit. **3** $t = (-B - \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit. **4** else $t = (-B + \sqrt{D})/2A$; if t is in $[t_0, t_1]$: hit. **5** else no hit. **6** at a hit: $\mathbf{p} = \mathbf{e} + t\,\mathbf{d}$, $\mathbf{n} = (\mathbf{p} - \mathbf{c})/R$. **7** colour $= \frac{1}{2}(\mathbf{n} + (1, 1, 1))$; no hit: the sky.

Six programs draw Ball B with a normal picture, and each has one slip. Name the step and guess the slip.

	what the picture shows	step	the slip
a	Python stops with "math domain error" on some pixels.		
b	A ball placed behind the camera shows up in the picture.		
c	Every pixel is sky, even the ones aimed straight at the ball.		
d	The ball comes out much smaller than it should.		
e	The middle of the ball is dark brown instead of light blue.		
f	Most of the ball comes out white.		

Part 4 · The ray meets a plane

Exercise 9

9 One ray, five planes

about 9 min

Put the ray into $(\mathbf{p}-\mathbf{a})\cdot\mathbf{n}=0$, one plane at a time: a hit, a far hit, a crossing behind the eye, a parallel ray, and a tilted plane.

The plane through $\mathbf{a}$ with normal $\mathbf{n}$ is $(\mathbf{p}-\mathbf{a})\cdot\mathbf{n}=0$. The ray meets it at $t=\dfrac{(\mathbf{a}-\mathbf{e})\cdot\mathbf{n}}{\mathbf{d}\cdot\mathbf{n}}$; check $\mathbf{d}\cdot\mathbf{n}$ before you divide.

The ray leaves $\mathbf{e}=(0,\,6,\,0)$ along $\mathbf{d}=(0,\,-2,\,-1)$, row 2 of Exercise 3. Five planes are around it.

plane	$\mathbf{a}$	$\mathbf{n}$	$(\mathbf{a}-\mathbf{e})\cdot\mathbf{n}$	$\mathbf{d}\cdot\mathbf{n}$	t	the point	hit?
floor, $y=0$	$(0,\,0,\,-6)$	$(0,\,1,\,0)$	-6	-2	3	$(0,\,0,\,-3)$	yes
back wall, $z=-10$	$(0,\,0,\,-10)$	$(0,\,0,\,1)$	_____	_____	_____	_____	_____
ceiling, $y=8$	$(0,\,8,\,0)$	$(0,\,-1,\,0)$	_____	_____	_____	_____	_____
side wall, $x=3$	$(3,\,0,\,0)$	$(-1,\,0,\,0)$	_____	_____	_____	_____	_____
tilted plane	$(0,\,0,\,-2)$	$(0,\,2,\,1)$	_____	_____	_____	_____	_____

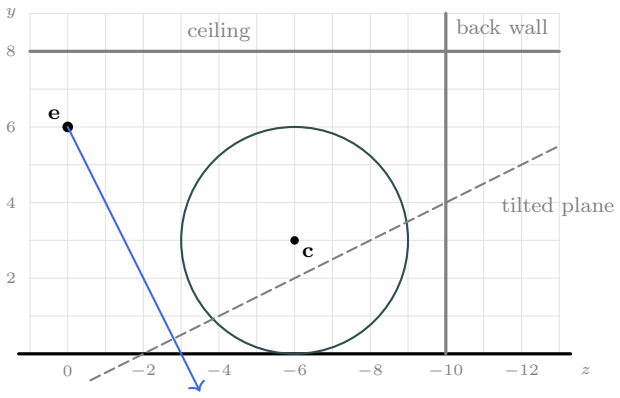
(a) Which plane does the eye see first? Mark all the crossings with $t>0$ on the ray in the side view below.

(b) Redo the tilted plane with $\mathbf{n}=(0,\,4,\,2)$, and then with $\mathbf{n}=(0,\,-2,\,-1)$. What happens to t, and why?

(c) The back wall's point lies 14 below the floor. Why does the formula report it anyway, and why does that do no harm here?

(d) Does this ray hit Ball B? (Exercise 3 already knows.)

(e) Swap the floor for a huge ball, radius 100, whose top touches the floor at $(0,\,0,\,-6)$. Without computing: does the ray meet it before or after $t=3$? Which rays would hit the flat floor but miss the huge ball?



Try it tonight optional, about 10 minutes, no marks

The ball of Exercise 6, drawn by your computer in text: one ray per character, `@` for a hit and `.` for a miss. Characters are about twice as tall as they are wide, so the window gets 40 columns and 20 rows.

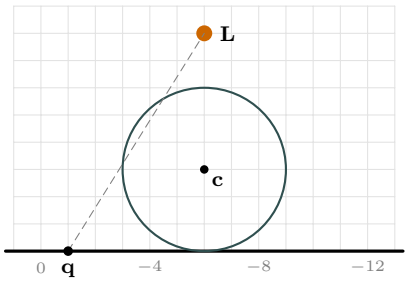
```
W, H = 40, 20                          # characters across and down
for j in range(H - 1, -1, -1):         # the top row first
    row = ''
    for i in range(W):
        x = -1 + 2 * (i + 0.5) / W     # the window runs from -1 to 1
        y = -1 + 2 * (j + 0.5) / H
        A = x * x + y * y + 1          # d = (x, y, -1)
        B = -10                        # 2 d.(e - c), with e - c = (0, 0, 5)
        C = 16                         # 25 - 9
        D = B * B - 4 * A * C
        row += '@' if D >= 0 else '.'
    print(row)
```

1. Run it. The disc has radius 0.75 on a window 2 tall: how many of the 20 rows should it cover, and how many does it?
2. Colour by the normal, one part only: at a hit, find t_-, the point $\mathbf{p} = t_- \mathbf{d}$ and the normal's y part, $n_y = p_y/3$. Print a character from `' .:-=+*%@'` chosen by the green part $\frac{1}{2}(n_y + 1)$: the brighter the green, the later in the string. Why does the ball now look lit from above?
3. Move the ball to (0, 0, 5), behind the eye. What does the first program print? Fix it with the interval rule.

10 Bonus: the ball's shadow ★

about 10 min

For the brave. Lecture 11's shadow ray meets today's sphere test: which floor points does Ball B hide from a lamp?



A lamp hangs at $\mathbf{L} = (0, 8, -6)$, 5 above the ball's centre. A floor point $\mathbf{q} = (0, 0, z)$ is in shadow when its shadow ray, from $\mathbf{q}$ along $\mathbf{d} = \mathbf{L} - \mathbf{q}$, meets the ball for some t between 0 and 1 ($t = 1$ is the lamp). Write $u = z + 6$.

(a) Show that $\mathbf{q} - \mathbf{c} = (0, -3, u)$ and $\mathbf{d} = (0, 8, -u)$, and so $A = 64 + u^2$, $B = -48 - 2u^2$ and $C = u^2$.

(b) Show that $D = 2304 - 64\,u^2$. (The u^4 terms cancel.)

(c) Which floor points on the line $x = 0$ are in shadow? Give the two ends as z values. Show that for these points both roots lie between 0 and 1.

(d) Check the ends with a right triangle: lamp, centre, and the point where an edge ray grazes the ball (3, 4, 5 again), and similar triangles down to the floor.

(e) At the contact point $(0, 0, -6)$ the roots are 0 and $\frac{3}{4}$. What does each mean? Is the contact point dark?

(f) Why must the roots lie between 0 and 1? Describe a lamp for which a floor point has $D > 0$ and is still lit.

Answer Key

Check your work *after* attempting everything. If an answer surprises you, that item is worth re-reading in the slides.

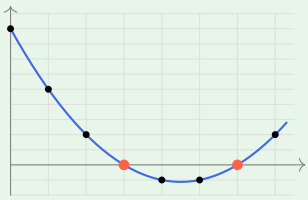
1.

point	$\mathbf{p}-\mathbf{c}$	number	verdict
$(0,3,-3)$	$(0,0,3)$	$9-9=0$	on (the front point)
$(0,4,-5)$	$(0,1,1)$	$2-9=-7$	inside
$(2,1,-4)$	$(2,-2,2)$	$12-9=3$	outside, just
$(-1,5,-4)$	$(-1,2,2)$	$9-9=0$	on
$(0,0,-6)$	$(0,-3,0)$	$9-9=0$	on (where it touches the floor)
$(0,3,-6)$	$(0,0,0)$	$0-9=-9$	inside: the centre

1a. The centre, $-9=-R^2$. A dot product of an arrow with itself is a squared length, never negative, so the number is at least $0-R^2$.

1b. Under C: $C=(\mathbf{e}-\mathbf{c})\cdot(\mathbf{e}-\mathbf{c})-R^2$ is exactly this number at the eye.

2. $\mathbf{p}(t)=(0,\ 6-t,\ -t)$ and $\mathbf{p}(t)-\mathbf{c}=(0,\ 3-t,\ 6-t)$. The numbers for $t=0,\dots,7$: 36, 20, 8, 0, −4, −4, 0, 8.



2b. It crosses zero at $t=3$ and $t=6$: the points $(0,3,-3)$, the front of the ball, and $(0,0,-6)$, where the ball touches the floor. The ray is inside for $3<t<6$, where the curve is below zero.

2c. $C=36$. $A=\mathbf{d}\cdot\mathbf{d}=2$ and $B=2\,\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})=2\,(0-3-6)=-18$, so the curve is $2t^2-18t+36$; at $t=1$: $2-18+36=20$, as in the table.

3.

	A	$\mathbf{d}\cdot(\mathbf{e}-\mathbf{c})$	B	C	D	verdict	t_-	t_+
1	1	−6	−12	36	$144-144=0$	graze	6	6
2	5	−12	−24	36	$576-720=-144$	miss		
3	6	−15	−30	36	$900-864=36$	hit	2	3
4	25	−30	−60	36	$3600-3600=0$	graze	1.2	1.2
5	30	−33	−66	36	$4356-4320=36$	hit	1	1.2

3a. C uses only the eye, the centre and R, never $\mathbf{d}$; every ray starts at the same eye. It is the eye's number from Exercise 1, 36. A program can compute it once per ball.

3b. Row 1 runs level at height 6 and touches the top, $(0,6,-6)$. Row 4 touches the lower front, $(0,\ 1.2,\ -3.6)$. The worked ray goes in at $(0,3,-3)$. Every ray from the eye between the two grazing rays meets the ball: they are the top and bottom edges of the ball in the picture.

3c. $6-2t=0$ at $t=3$: the point $(0,\ 0,\ -3)$, just in front of the ball, which touches the floor at $z=-6$.

3d. It goes in at $(2,5,-5)$ and out at $(2.4,\ 4.8,\ -6)$, a short chord that cuts a thin slice off the ball's side. Its pixel lies just inside the ball's outline in the picture.

4.

	first hit	the ray starts ...
1	$t_+=2$ (case 2)	inside the ball
2	none (case 3)	past the ball: the ball is behind it
3	none (case 3)	outside, but the ball is beyond t_1
4	$t_+=6$	outside; the interval skips the way in
5	$t_-=1.2$, a graze	outside, the ray just touches
6	$t_+=3$	on the surface
7	none, before any case	anywhere; the line misses
8	$t_-=2$: the interval includes its end	outside

4b. The start is on the ball, so 0 is a root. After rounding it comes out as a tiny number, sometimes just above 0. With $t_0=0$ the ray would find its own starting point and report a hit there, lecture 11's shadow acne. A small t_0 skips it.

4c. D alone catches row 7. Rows 2 and 3 have $D>0$ and two real roots; only the interval throws them out.

5.

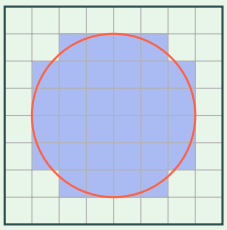
	A	B	C	D	t_-	t_+	first hit
1	8	-36	36	144	1.5	3	1.5, the same point $(0, 3, -3)$
2	2	18	36	36	-6	-3	none
3	2	-10	8	36	1	4	1, the same point $(0, 3, -3)$
4	2	-6	0	36	0	3	3 with a small t_0: $(0, 0, -6)$
5	2	-2	-4	36	-1	2	$t_+ = 2$: $(0, 0, -6)$
6	2	18	36	36	-6	-3	none

5a. A and D stay; B and C change, and both roots drop by the number of steps, so they still name the same two points. $t_+ - t_- = \sqrt{D}/A$ is the part of the ray inside the ball, counted in steps of $\mathbf{d}$. Sliding the eye along the line does not change that part, and A does not change, so D cannot either.

5b. Reversing $\mathbf{d}$ flips the sign of $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c})$; moving the ball to the other side of the eye flips $\mathbf{e} - \mathbf{c}$ instead. Either way only B changes sign, and D uses B^2. The roots come out negative: both crossings are behind the eye. A program that only tests $D \geq 0$ would draw the ball behind the camera, exactly where the real one was.

5c. Row 4 starts on the ball ($C = 0$): with a small t_0 the first hit is t_+, case 2. Row 5 starts inside ($C < 0$): case 2.

6.



6a. $\mathbf{e} - \mathbf{c} = (0, 0, 5)$, so $A = x^2 + y^2 + 1$, $\mathbf{d} \cdot (\mathbf{e} - \mathbf{c}) = -5$ and $B = -10$, and $C = 25 - 9 = 16$. Then $D = 100 - 64\,(1 + x^2 + y^2) = 36 - 64\,(x^2 + y^2)$.

6b. $x^2 + y^2 \leq 36/64 = 0.75^2$: the circle of radius 0.75 around the window's centre.

6c. 32 pixels: rows of $0, 4, 6, 6, 6, 6, 4, 0$ (red circle: radius 0.75).

6d. Eye to centre 5, centre to the grazing point 3 (a radius, perpendicular to the grazing ray), so eye to the grazing point 4. The grazing ray moves 3 sideways for every 4 ahead: at the window, 1 ahead, it is 0.75 from the middle.

6e. Each pixel is all ball or all sky, so the edge follows the pixel squares. 16×16 pixels make the steps half as big, but a staircase is still a staircase. Lecture 15 smooths it with several rays per pixel.

6f. Both have length 5. $(0, 3, -4)$: $\hat{y} = 0.6$, $a = 0.8$, colour $0.2\,(1, 1, 1) + 0.8\,(0.5, 0.7, 1) = (0.6,\ 0.76,\ 1)$. $(0, -3, -4)$: $\hat{y} = -0.6$, $a = 0.2$, colour $(0.9,\ 0.94,\ 1)$: paler. The lower a ray points, the smaller a and the more white it gets, so the sky is palest near the ground.

7.

ray	$\mathbf{p}$	$\mathbf{p} - \mathbf{c}$	$\mathbf{n}$	colour
row 1	$(0, 6, -6)$	$(0, 3, 0)$	$(0, 1, 0)$	$(0.5,\ 1,\ 0.5)$
row 3	$(2, 4, -4)$	$(2, 1, 2)$	$(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$	$(\frac{5}{6},\ \frac{2}{3},\ \frac{5}{6}) \approx (0.83,\ 0.67,\ 0.83)$
row 4	$(0,\ 1.2,\ -3.6)$	$(0,\ -1.8,\ 2.4)$	$(0,\ -0.6,\ 0.8)$	$(0.5,\ 0.2,\ 0.9)$
row 5	$(2, 5, -5)$	$(2, 2, 1)$	$(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$	$(\frac{5}{6},\ \frac{5}{6},\ \frac{2}{3}) \approx (0.83,\ 0.83,\ 0.67)$

	$\mathbf{n}$	length	faces	correct picture?
A	$(0,\ 0.8,\ 0.6)$	1	up and toward us	yes
B	$(-0.6,\ 0,\ 0.8)$	1	left and toward us	yes
C	$(0,\ 0,\ 0)$	0	nowhere	no
D	$(0,\ -0.8,\ -0.6)$	1	down and away from us	no
E	$(1,\ 1,\ 1)$	$\sqrt{3} \approx 1.73$	up, right, toward us	no
F	$(0.8,\ 0,\ 0.6)$	1	right and toward us	yes

7a. C, D and E. C: a zero normal, one that was never set, or $\mathbf{p} - \mathbf{c}$ worked out at the centre. D: the far side of the ball, so the program tried t_+ first. E: not a unit normal, so the program did not divide by R (its parts were up to 3).

7b. Near the middle of the picture the ray runs along $-z$. A surface the eye can see faces back toward the eye, so its normal has a positive z part and its blue is above 0.5. D's normal points along the ray, away from the eye: that is the back of the ball.

8.

	step	the slip
a	2	the square root is taken although $D < 0$: the test is missing, or comes after the root
b	3–5	no interval test: every $D \geq 0$ counts as a hit, even with both roots negative
c	1	B written with $\mathbf{c} - \mathbf{e}$ instead of $\mathbf{e} - \mathbf{c}$: every root changes sign, so every hit lies behind the eye and the interval throws it out
d	1	C subtracts R instead of R^2: $45 - 3 = 42$ instead of 36, the program's ball has radius $\sqrt{3}$
e	3	t_+ tried first: the program colours the back of the ball, whose normals face away: blue near 0, about (0.5, 0.28, 0.05)
f	6	$\mathbf{n}$ not divided by R: parts up to 3, clipped to 1 in the colour, so most pixels come out white

9.

plane	$(\mathbf{a} - \mathbf{e}) \cdot \mathbf{n}$	$\mathbf{d} \cdot \mathbf{n}$	t	the point	hit?
back wall	-10	-1	10	(0, −14, −10)	yes, but 14 below the floor
ceiling	-2	2	-1	(0, 8, 1)	no: behind the eye
side wall	-3	0	—	—	no: parallel
tilted	-14	-5	2.8	(0, 0.4, −2.8)	yes

9a. The tilted plane, at $t = 2.8$; then the floor at 3 and the back wall at 10.

9b. t stays 2.8. Doubling $\mathbf{n}$ doubles the top and the bottom of the fraction; flipping it flips both. The plane is the same, so its crossing is the same: $\mathbf{n}$ need not be a unit normal here.

9c. The formula knows only the one plane, and a plane goes on forever, below the floor too. The eye sees the nearest hit, and the floor and the tilted plane come first.

9d. No: row 2 of Exercise 3 has $D = -144$.

9e. After. Away from $(0, 0, -6)$ the huge ball's surface curves down below the floor, so the ray reaches $y = 0$ first and the ball a little later ($t \approx 3.02$). Nearly level rays meet the flat floor far away but pass above the huge ball's horizon.

10a. $\mathbf{q} - \mathbf{c} = (0,\ -3,\ z + 6) = (0, -3, u)$, $\mathbf{d} = (0,\ 8,\ -6 - z) = (0, 8, -u)$. $A = 64 + u^2$, $\mathbf{d} \cdot (\mathbf{q} - \mathbf{c}) = -24 - u^2$ so $B = -48 - 2u^2$, and $C = 9 + u^2 - 9 = u^2$.

10b. $D = (48 + 2u^2)^2 - 4\,(64 + u^2)\,u^2 = 2304 + 192u^2 + 4u^4 - 256u^2 - 4u^4 = 2304 - 64u^2$.

10c. $D \geq 0$ when $u^2 \leq 36$: $-6 \leq u \leq 6$, so $-12 \leq z \leq 0$, a stretch 12 long centred under the ball; its near end is right below the eye. The roots are $(48 + 2u^2 \pm \sqrt{D})/(128 + 2u^2)$. Neither is negative, because $\sqrt{D} \leq 48$ (the smaller is 0 only at the contact point, $u = 0$); both are below 1, because $48 + 2u^2 + \sqrt{D} \leq 96 + 2u^2 < 128 + 2u^2$.

10d. Lamp to centre 5, centre to the grazing point 3, so lamp to the grazing point 4. The edge ray moves 3 sideways for every 4 down; from the lamp to the floor is 8 down, so 6 sideways: $z = -6 \pm 6$. The edge rays touch the ball at (0, 4.8, −3.6) and (0, 4.8, −8.4), at $t = 0.6$.

10e. $t = 0$ is the contact point itself, on the ball. $t = \frac{3}{4}$ is where the shadow ray leaves the ball again, (0, 6, −6), the top. The ball stands between the point and the lamp, so it is dark, even with a small t_0.

10f. $t = 1$ is the lamp. A root above 1 is a crossing beyond the lamp, which cannot block its light. Example: a low lamp between a floor point and the ball, such as $\mathbf{L} = (0, 1, -10)$ and $\mathbf{q} = (0, 0, -14)$. The line from $\mathbf{q}$ through the lamp runs on into the ball, so $D > 0$, but both roots are above 1 and the point is lit.

T1. About $1.5/2 \times 20 = 15$ rows; it covers 14. The row centres sit at $y = \pm 0.05, \pm 0.15, \ldots, \pm 0.95$, and only those with $|y| < 0.75$ reach the disc; the rows at ± 0.75 graze it exactly on the edge and every column misses by a hair.

T2. At a hit, $t_- = (10 - \sqrt{D})/(2A)$, $p_y = t_- y$ and $n_y = p_y/3$. The top of the ball faces up (n_y near 1) and gets the brightest characters, the bottom faces down and gets the darkest, like a ball lit from above. This is only the green part of the lecture's colour, not real lighting (lecture 16).

T3. The same ball: the program tests $D \geq 0$ only, and D does not change when the ball moves to the other side of the eye. Fix: at a hit, also require $t_- \geq 0$ (or $t_+ \geq 0$).