Finger Exercises 16

Point Lights and Lambertian Shading

≈ 70 minutes, plus a bonus

CS 453 — Computer Graphics

You need a pencil and a ruler. Every number was chosen to come out by hand. The letters are the lecture's: the shading point $\mathbf{x}$; the lamp's position $\mathbf{p}$ and its intensity I; the distance $r = \|\mathbf{p} - \mathbf{x}\|$; the unit arrows $\mathbf{n}$, out of the surface, and $\mathbf{l} = (\mathbf{p} - \mathbf{x})/r$, toward the lamp; the irradiance E, the light that arrives; the reflectance R; the material's factor $k = R/\pi$; and the reflected light L_r, which becomes the pixel's colour. In the code, a light's *share* is the colour of the light that $\mathbf{x}$ sends to the eye because of that one light. Ambient light has an amount I_a, and a material sends the fraction R of it to the eye. The letter R is taken, so a ball's radius is given as a plain number. Nothing here is an assignment task; the sheet is Lecture 16 only.

Short on time? Do Exercises 2, 4, 8 and 12 (about 30 minutes): the cosine by counting, the whole chain from positions to E, whose number is whose, and the loop run by hand.

Part 1 · How much light arrives

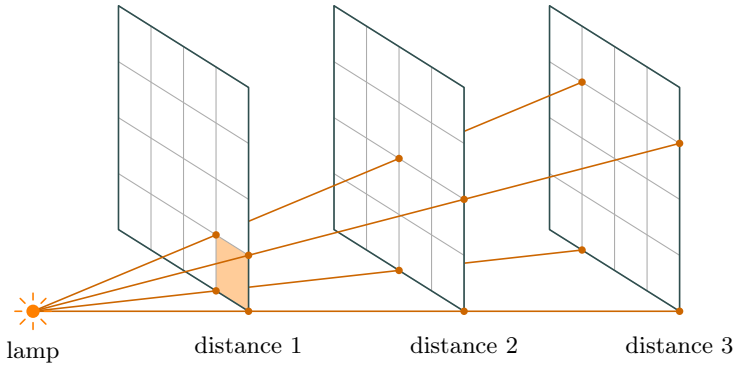
Exercises 1–6

1 Count the tiles

about 5 min

The inverse square, by counting: the same light lands on more tiles.

A lamp shines a beam through a square hole onto a tiled wall. The picture shows the wall at three distances from the lamp. At distance 1 the beam lights exactly one tile. The four orange lines are the edges of the beam, and a dot marks the place where an edge meets a wall.



distance r	tiles lit	light on one tile
1	1	36
2	_____	_____
3	_____	_____
6	_____	_____

(a) Shade the tiles that the beam lights on the walls at distances 2 and 3, and fill the column "tiles lit". Distance 6 needs no picture.

(b) The beam carries 36 units of light and loses none on the way. How much lands on *one* tile at each distance? Fill the last column.

(c) Write the light on one tile as a formula in r. The lecture writes $E = I/r^2$. Which of your numbers plays the part of I, and why does it not change with the distance?

(d) You read with the page 50 cm from a lamp. Then you move your chair, and the page is 2 m from the lamp. How many such lamps must be switched on, side by side, to give the page as much light as before?

(e) A floodlight shines equally in all directions, with $I = 13.6$ watts per steradian. All directions together make $4\pi \approx 12.6$ steradians. How many watts of light does it send out in all? What is the irradiance on a page that faces it from 1 m away, and from 2 m?

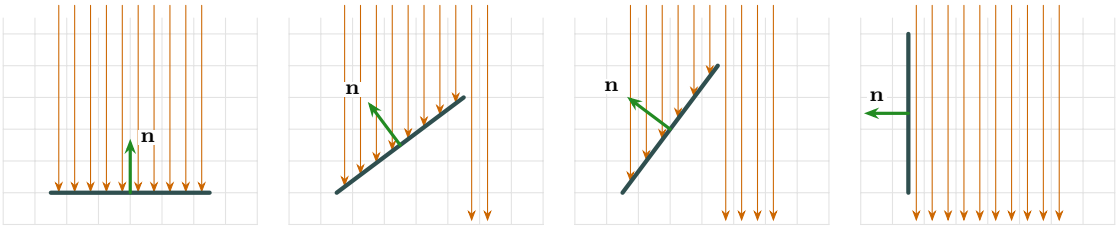
(f) Sunlight at noon delivers about 1360 W/m^2. How close to the floodlight must the page be held to get as much? (How many times more light is that than at 1 m?)

2 Count the rays

about 6 min

Lambert's cosine law, by counting: a tilted surface is hit by fewer rays. Then the dot product gives the count with no picture at all.

A stick stands for a small flat surface, seen from the side. Its green arrow, the unit normal $\mathbf{n}$, marks its outer side. One beam falls straight down on four sticks: ten rays, half a unit apart. Every stick is 5 long (stick 2 goes 4 across and 3 up). Arrows are pairs (across, up); the one toward the light is $\mathbf{l} = (0,\ 1)$.



stick 1 stick 2 stick 3 stick 4

stick	rays that land, of 10	$\mathbf{n}$	$\mathbf{n}\cdot\mathbf{l}$, $\mathbf{l} = (0,\ 1)$	$\mathbf{n}\cdot\mathbf{l}'$, $\mathbf{l}' = (0.6,\ 0.8)$	rays you predict, of 10
1	10	$(0,\ 1)$	1	_____	_____
2	_____	$(-0.6,\ 0.8)$	_____	_____	_____
3	_____	$(-0.8,\ 0.6)$	_____	_____	_____
4	_____	$(-1,\ 0)$	_____	_____	_____

(a) Count the rays that land on each stick, and fill the column $\mathbf{n}\cdot\mathbf{l}$. What do you notice? Why may the count stand for the irradiance E on a stick?

(b) The light now comes from the upper right: the unit arrow toward it is $\mathbf{l}'$. Draw nothing. Fill the last two columns: the dot product, and the rays that you expect on each stick's outer side. What has happened to stick 3? What does the minus sign of stick 4 say?

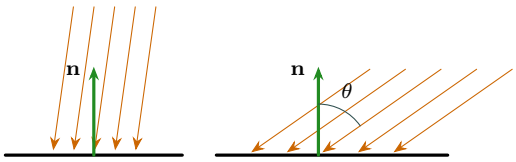
(c) Stick 2 again, with the light from above. A student takes the arrow to the lamp as it comes, $(0,\ 20)$, and does not make it one unit long. Which "cosine" comes out? How do you know at once that it is wrong?

3 Summer and winter

about 5 min

A directional source, $E = H\cos\theta$. For the sun, which matters: its distance, or the tilt of the ground?

The sun is a directional source: a whole city gets the same $H = I/r^2$ and the same $\mathbf{l}$. From June to December two things change: the sun's distance, and its height at noon. The numbers are for Lahore, with no air in the way.



late June late December

	late June	late December
the sun's distance r	152 million km	147 million km
$H = I/r^2$	1320 W/m^2	1410 W/m^2
noon sun, θ from straight up	8°	55°
$\cos\theta$	0.99	0.57
E on flat ground at noon	_____	_____

(a) In December the sun is nearer than in June. By what percentage? So by about what percentage should H be larger? (The roof slide: E goes as $1/r^2$, so it changes by about twice as much.) Check with the table.

(b) Fill the last row: E on flat ground at noon, in June and in December.

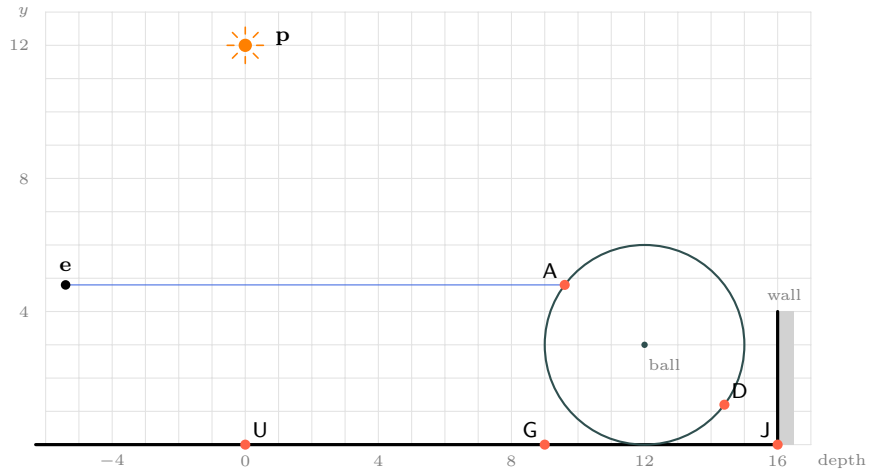
(c) The nearer sun gives the colder season. Which of the lecture's two factors makes summer and winter: the distance, or the tilt?

(d) A solar panel on a roof, at noon in December. Which way must its normal point to collect the most light? How much does it collect then? Compare with the flat ground in June.

4 From positions to E

about 8 min

The whole chain for one lamp: $\mathbf{p} - \mathbf{x}$, *then* r, *then* $\mathbf{l}$, *then* $\mathbf{n} \cdot \mathbf{l}$, *then* $E = \max(0,\ \mathbf{n} \cdot \mathbf{l})\, I/r^2$.



The courtyard comes back in Exercises 5, 9 and 12. The side view shows every point with $x = 0$ at its true size: $-z$ runs to the right (the depth) and y up.

lamp $\mathbf{p} = (0,\ 12,\ 0)$, $I = 144\pi$
floor $y = 0$
wall $z = -16$, 4 high
ball centre $(0,\ 3,\ -12)$, radius 3

K lies outside the slice: at the foot of the wall, 15 to the right of J.

	the point $\mathbf{x}$	its surface, and $\mathbf{n}$	$\mathbf{p} - \mathbf{x}$	r	$\mathbf{l}$	$\mathbf{n} \cdot \mathbf{l}$	E
	G $= (0,\ 0,\ -9)$	floor, $(0,\ 1,\ 0)$	$(0,\ 12,\ 9)$	15	$(0,\ 0.8,\ 0.6)$	0.8	$0.512\,\pi$
1	U $= (0,\ 0,\ 0)$	floor, $(0,\ 1,\ 0)$					
2	A $= (0,\ 4.8,\ -9.6)$	ball, $(0,\ 0.6,\ 0.8)$					
3	D $= (0,\ 1.2,\ -14.4)$	ball, $(0,\ -0.6,\ -0.8)$					
4	J $= (0,\ 0,\ -16)$	floor, $(0,\ 1,\ 0)$					
5	J $= (0,\ 0,\ -16)$	wall, $(0,\ 0,\ 1)$					
6	K $= (15,\ 0,\ -16)$	wall, $(0,\ 0,\ 1)$					

(a) Before any arithmetic: which of the floor points U, G and J gets the most light from the lamp, and which the least? Then fill the table. Write each E as a number times π, as in the worked row: $144\pi \times 0.8/15^2 = 0.512\,\pi$.

(b) U and A get the same E. Which two things do these points have in common?

(c) Rows 2 to 5 have the same $\mathbf{l}$. Draw the line from the lamp to J in the picture. What does it pass through on its way?

(d) At J the floor and the wall meet: the same point, the same lamp, the same r. Which of the two surfaces gets more light there, and why? A room is painted in one colour all over. Why can you still see its corners?

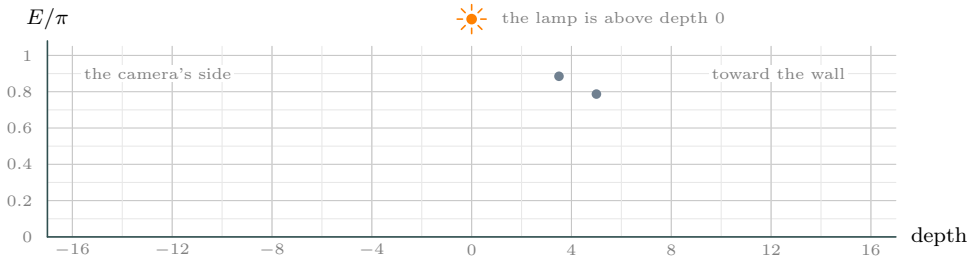
(e) D gets no light: the max says that the lamp is behind the surface there. J gets light, says the formula. Look at your line from (c). Would J be lit in a real courtyard? What does the formula not know?

(f) Only in row 6 does $\mathbf{p} - \mathbf{x}$ have no zero in it. Check that your $\mathbf{l}$ for K has length 1.

5 The pool of light

about 5 min

Distance and tilt together along one floor: why the light under a lamp fades so fast.



depth	r	E/π
0 (U)	12	
3.5	12.5	0.885
5	13	0.787
9 (G)	15	
16 (J)	20	

(a) Copy E/π for the floor at U, G and J from Exercise 4 into the small table, and mark all five points in the plot. The floor on the camera's side gets the same light at the same distance from U: mark those points too, and draw a smooth curve.

(b) From U to G the light falls to about half. Two factors do it: the distance, $(12/15)^2$, and the tilt, $12/15$. Which of the two costs more? Find the same two factors for J.

(c) Both factors are powers of one number: $\cos\theta = 12/r$. Write E on this floor with $\cos\theta$ alone. Check your formula at depth 3.5, where $\cos\theta = 0.96$.

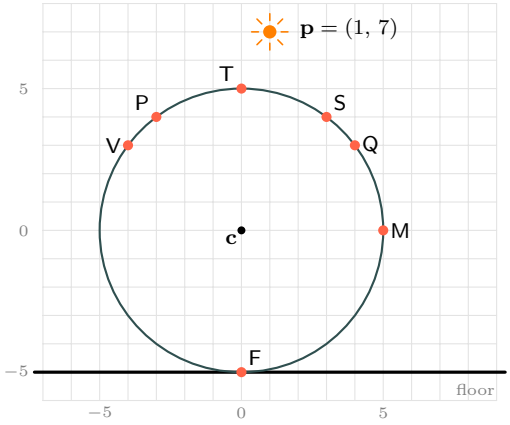
(d) Read from your curve: how far from U does the floor get half of U's light? Compare that distance with the lamp's height.

(e) A second, equal lamp is hung at the same height above depth -18. With both lamps on, what is E on the floor half way between them, at depth -9? Compare it with E at U under one lamp alone.

6 In front, or behind?

about 6 min

The sign of $\mathbf{n}\cdot\mathbf{l}$ says on which side of a surface the lamp is. Around a ball that gives the edge of the light.



A slice through a big ball and a lamp that hangs low over it. The ball's centre $\mathbf{c}$ is the origin of the grid and its radius is 5. At a point $\mathbf{x}$ of the ball, the arrow $\mathbf{u} = \mathbf{x} - \mathbf{c}$ points straight out of the ball. It is 5 long, so $\mathbf{n} = \mathbf{u}/5$. So $\mathbf{u}\cdot(\mathbf{p}-\mathbf{x})$ is $5r$ times $\mathbf{n}\cdot\mathbf{l}$: the same sign, and no square root.

	$\mathbf{u}$	$\mathbf{p}-\mathbf{x}$	$\mathbf{u}\cdot(\mathbf{p}-\mathbf{x})$	the lamp is
T	(0, 5)	(1, 2)	$0+10=10$	in front
S	(3, 4)			
P	(−3, 4)			
Q	(4, 3)			
M	(5, 0)			
V	(−4, 3)			
F	(0, −5)			

(a) Fill the table. Positive: the lamp is in front of the surface there. Negative: it is behind. Zero: the light only grazes the surface.

(b) Lay your ruler from the lamp to P, then from the lamp to Q, and draw both lines. What do they do at the ball? Which angle do they make with $\mathbf{u}$ there?

(c) Go over the part of the ball's outline that this lamp cannot light with a thick pencil line. Is it more than half of the outline, or less?

(d) Now take the lamp very far away in the same direction, like the sun. Then $\mathbf{l}$ is the same arrow at every point, along (1, 7), and the sign of $\mathbf{u}\cdot(1,\ 7)$ decides. Which of the seven points are lit now? Where does the border of the light run?

(e) At F, $\mathbf{n}\cdot\mathbf{l} \approx -1$ and $r^2 = 145$. What does $E = I\,(\mathbf{n}\cdot\mathbf{l})/r^2$ give there without the max? What would that do to a pixel that a second lamp lights?

Part 2 · What the surface sends back

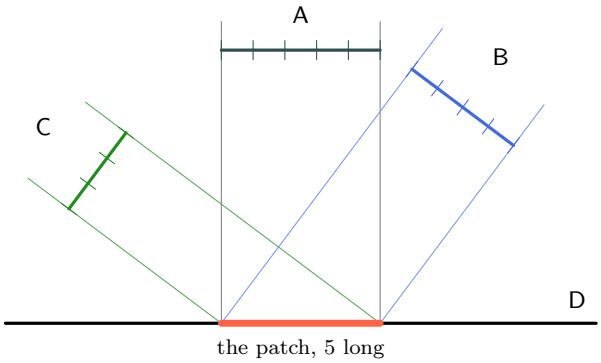
Exercises 7–9

7 Where the π comes from

about 7 min

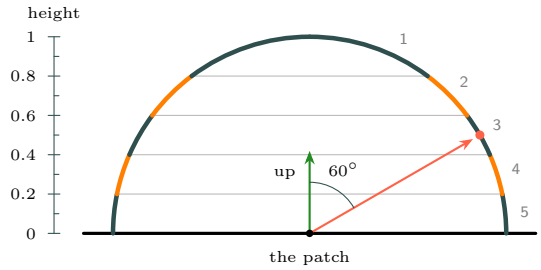
Equal brightness is not equal light: a slanted direction gets less of it. Added up over all directions, the cosines come to π, not 2π.

A small Lambertian patch, 5 long, lies on the floor; you see it from the side. It shows every eye the same brightness, L_r. Each eye is far away, so its two sight lines, one from each end of the patch, are parallel. The bar across them is that eye's ruler: it measures how wide the patch looks from there.



eye	θ, from straight up	$\cos\theta$	the patch looks … wide	light toward that eye
A	0°	1	5	10
B	36.9°	0.8	_____	_____
C	53.1°	0.6	_____	_____
D	90°	0	_____	_____

Every direction above the patch is a point of a dome of radius 1 around it. The second picture cuts the dome through its middle. Five bands of equal height, 0.2, cover it. Archimedes: on a sphere, bands of equal height have equal area. Each of these has $2\pi \times 0.2 = 0.4\,\pi$, the small cap at the top and the wide ring at the floor alike.



band	$\cos\theta$ at its middle height	its area	area × cosine
1, the cap	0.9	$0.4\,\pi$	$0.36\,\pi$
2	_____	$0.4\,\pi$	_____
3	_____	$0.4\,\pi$	_____
4	_____	$0.4\,\pi$	_____
5, at the floor	_____	$0.4\,\pi$	_____
all five		$2\,\pi$	_____

(a) Count the units on the bars of A, B and C, and fill the column "looks … wide". The eye at D is level with the floor. Check that each width is $5\cos\theta$.

(b) The light that goes toward an eye is the brightness times the width that the eye sees. The brightness is L_r for every eye, and 10 units of light go toward A. Fill the last column. Then finish the sentence: the light toward a direction θ from straight up is $L_r \times$ _____.

(c) Now the dome. The red arrow leaves the patch 60° from straight up. Read the height of its red dot off the scale, and compare it with $\cos 60°$. So on this dome the cosine of a direction is its _____.

(d) Fill the second table: the cosine at the middle height of each band, then area × cosine. Add up the last column. Which number comes out?

(e) If every direction got the full L_r, the sum would be the dome's area, 2π. So what is the mean cosine over the dome? Now cut the dome into two bands only, with heights 0 to 0.5 and 0.5 to 1, and add again. Does the number of bands matter?

(f) So a patch of brightness L_r sends back $\pi\,L_r$ in all. A plaster wall sends back 188 units of light. Which brightness L_r does every eye see? (Exercise 8 starts with this wall.)

8 Whose number is it?

about 7 min

$L_r = k\,E$, with $k = R/\pi$. The lighting sets E; the material sets k.

Light arrives at four flat surfaces: $E = 314$ units at each. (That is 100π, so the division by π comes out by hand.) A Lambertian surface sends back the fraction R of it, at the same brightness L_r toward every eye. By Exercise 7 that light is $\pi\,L_r$ in all. So $\pi\,L_r = R\,E$, and every eye sees $L_r = (R/\pi)\,E$.

surface	R	sent back, $R\,E$	absorbed, $(1-R)\,E$	toward one eye, L_r	$k = L_r/E$
plaster	0.6	188	126	60	0.19
fresh snow	0.9	_____	_____	_____	_____
brick	0.3	_____	_____	_____	_____
tarmac	0.1	_____	_____	_____	_____

Now the plaster wall alone. It starts with $E = 314$, $k = 0.19$ and $L_r = 60$. One thing changes at a time. Write the new value, or "same".

	what changes	E	k	L_r
1	A second, equal lamp is switched on beside the first.	_____	_____	_____
2	The one lamp is moved twice as far away.	_____	_____	_____
3	The wall is turned until $\mathbf{n}\cdot\mathbf{l}$ falls from 1 to 0.5.	_____	_____	_____
4	The eye walks to the other end of the courtyard.	_____	_____	_____
5	The wall is repainted with $R = 0.3$.	_____	_____	_____
6	The lamp is carried round to the back of the wall.	_____	_____	_____
7	Rows 1 and 5 together: the darker paint, and two lamps.	_____	_____	_____

(a) Fill both tables.

(b) Which rows of the second table moved E, and which moved k? Finish the two sentences: "E belongs to _____. k belongs to _____."

(c) Row 7 ends with the L_r of the start. A camera records only L_r. Which two walls can it not tell apart?

(d) In the first table, add "sent back" and "absorbed" in each row. What comes out, and why must it? What would $R = 1.2$ mean?

(e) No reflectance is above 1. So what is the largest k that a Lambertian surface can have? Which of the four surfaces comes closest?

(f) Snow sends back 90% of the light that arrives. Is that number its R or its k? Does $k = 0.29$ mean that 29% of the light goes to the eye?

Try it tonight: with a torch optional, about 3 minutes, no marks

You need your phone's torch, this sheet and a dark room.

1. Hold the sheet 20 cm from the torch, facing it. Tilt it by about 60°. Then hold it facing the torch again, at 40 cm. Which darkens the paper more, the tilt or twice the distance? By how much should each?
2. Look at the lit paper from the left, from the right and from above. Does its brightness change? Now light a glossy magazine cover in the same way. What moves when your head moves? Which of the two surfaces is close to Lambertian?

9 Same paint? about 5 min

One reflectance for each channel: under a white lamp, every pixel of a surface is its R times one number.

The courtyard has three materials: the sandstone **floor**, $R = (0.8,\ 0.7,\ 0.6)$; the brick **wall**, $R = (0.6,\ 0.3,\ 0.2)$; and the glazed **ball**, $R = (0.2,\ 0.4,\ 0.8)$. The lights are neutral, so every light's share is R times a number, and so is their sum: a pixel's colour is $s \times R$, with the same s in all three channels. Here are six pixels of one picture.

pixel	its colour	surface	s	pixel	its colour	surface	s
1	(0.4, 0.35, 0.3)	_____	____	4	(0.04, 0.08, 0.16)	_____	____
2	(0.24, 0.48, 0.96)	_____	____	5	(1.2, 1.05, 0.9)	_____	____
3	(0.3, 0.15, 0.1)	_____	____	6	(0.25, 0.275, 0.35)	_____	____

(a) Divide a pixel's colour by an R, channel by channel. If the same number comes out three times, you have found the surface, and the number is s. Fill the table. One pixel fits no surface.

(b) Pixels 2 and 4 show the same ball, one six times as bright as the other. The paint is the same. So what differs between the two hits?

(c) Pixel 5 has numbers above 1. Which three bytes does lecture 15 store for it? (A number above 0.999 becomes 0.999; the byte is the whole part of 256 times the number.) What has happened to the colour of the sandstone?

(d) The pixel that fits no surface lies on the edge of the ball, with the floor behind it. Show that it is the mean of pixel 1 and $(0.1,\ 0.2,\ 0.4)$, the ball with $s = 0.5$. Which lecture mixes two surfaces in one pixel?

(e) The neutral lamp is changed for an orange one: its three intensities are $144\pi \times (1,\ 0.5,\ 0)$. At a point where E was π in every channel, it is now $\pi\,(1,\ 0.5,\ 0)$. Find L_r there for each of the three materials. What has become of the blue ball, and why?

Part 3 · Shading in the ray tracer Exercises 10–12

```
class PointLight(Light):  # intensity, position
    def illuminate(self, ray, rec):
        x = ray.evaluate(rec.t)
        r = (self.position - x).length()
        l = (self.position - x) / r
        n = rec.normal
        v = -ray.d / ray.d.length()
        E = max(0, n.dot(l)) * self.intensity / r**2
        k = rec.surface.material.evaluate(l, v, n)
        return k * E

class Lambertian(Material):     # has albedo
    def evaluate(self, l, v, n):
        return self.albedo / math.pi
```

```
class AmbientLight(Light):     # intensity: Ia
    def illuminate(self, ray, rec):
        R = rec.surface.material.albedo
        return R * self.intensity

def trace(ray):
    rec = scene.hit(ray)
    if rec.t == math.inf:
        a = 0.5 * (ray.d.y + 1.0)
        return Vec3(1.0, 1.0, 1.0).lerp(
            Vec3(0.5, 0.7, 1.0), a)
    c = Vec3(0,0,0)
    for light in lights:
        c = c + light.illuminate(ray, rec)
    return c
```

10 Who calls whom? about 3 min

The lecture's code, in the box above: which function calls which, and what comes back.

(a) One pixel's ray hits the ball. The scene's list holds an ambient light and two lamps. For this pixel, how often is each called: `scene.hit`, `illuminate`, `evaluate`? Which light never calls `evaluate`?

(b) Every `illuminate` returns its light's share: the colour of the light that _____ sends to _____ because of _____. What does `trace` do with the three shares?

11 Who does what?

about 5 min

A light works out its own share and asks the material for k; one loop adds the shares.

Each thing below is stored or worked out in one of five places. Write its letter: **P** the point light, **A** the ambient light, **M** the material, **H** the hit record, **T** `trace`.

the lamp's position $\mathbf{p}$	____	the reflectance R	____	the number of lights	____
the distance r	____	the factor k	____	the background colour	____
the cosine, $\mathbf{n} \cdot \mathbf{l}$	____	the lamp's intensity I	____	the surface that was hit	____
the irradiance E	____	the intensity I_a	____	the normal $\mathbf{n}$ at the hit	____

```
class DirectionalLight(Light):        # has H and l
    def illuminate(self, ray, rec):
        n = rec.normal
        v = -ray.d / ray.d.length()
        E = ______________________________
        k = rec.surface.material.evaluate(
                ___________, v, n)
        return ___________
```

(a) Fill the twelve blanks of the table.

(b) A third kind of light: a directional one, like the sun. It stores its normal irradiance H, a colour, and the unit arrow $\mathbf{l}$ toward the source. Fill the three blanks of its `illuminate`.

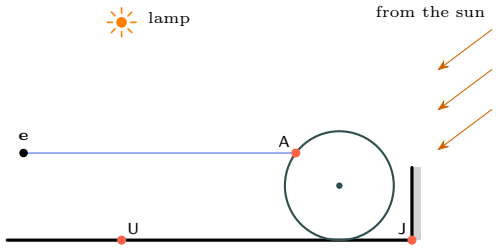
(c) Which three lines of the point light have no partner in the directional light, and why? To put the sun into a scene, which other code must change?

(d) The cosine sits in the light's code, not in the material's. Why is that the right place for it?

12 Be the loop

about 8 min

`trace` by hand, for four pixels and three kinds of light: add every light's share.



The courtyard at dusk. Its list of lights has three entries:
ambient $I_a = 0.2$
the lamp $\mathbf{p} = (0,\ 12,\ 0)$, $I = 144\pi$
the sun low behind the wall: $\mathbf{l}_s = (0,\ 0.6,\ -0.8)$, $H = 0.5\,\pi$
All three are neutral, so every share is a number times the surface's R: the ambient share is $0.2\,R$, the lamp's is $(E/\pi)\,R$, and the sun's is $0.5\,\max(0,\ \mathbf{n} \cdot \mathbf{l}_s)\,R$. The sky is $(0.5,\ 0.7,\ 1)$.

the pixel sees	$\mathbf{n}$	ambient share	the lamp: E/π, Exercise 4	$\mathbf{n} \cdot \mathbf{l}_s$	the sun: its share	the sum, s
the floor at U	$(0,\ 1,\ 0)$	0.2	1	0.6	0.3	1.5
the ball at A	$(0,\ 0.6,\ 0.8)$	_____	_____	_____	_____	_____
the wall at J	$(0,\ 0,\ 1)$	_____	_____	_____	_____	_____
the floor at J	$(0,\ 1,\ 0)$	_____	_____	_____	_____	_____
the sky	no hit: the pixel's colour is _____					

(a) First one hit in full. The eye is at $\mathbf{e} = (0,\ 4.8,\ 5.4)$. A ray with $\mathbf{d} = (0,\ 0,\ -1)$ comes back with the record: the ball, $t = 15$, $\mathbf{n} = (0,\ 0.6,\ 0.8)$. Run the point light's `illuminate` line by line: $\mathbf{x}$, r, $\mathbf{l}$, $\mathbf{v}$, E, k, and what it returns. Which of these does the ball never use?

(b) Fill the table: each share as a number that multiplies R, and the sum s.

(c) Turn the sums for the ball at A and the floor at U into colours, $s \times R$. You have met both in Exercise 9. Which pixels are they?

(d) A student's light classes have no max. Work out the sun's share and the sum again for the wall at J. Is the wall brighter or darker than with the sun switched off?

(e) The sun stands behind the wall, which is 4 high: its shadow covers about 5 units of floor in front of it. And the ball stands between J and the lamp (Exercise 4). Which shares in the row "the floor at J" would a real courtyard not have? What should s be there? What does the program lack?

Try it tonight: in Python optional, about 10 minutes, no marks

Lecture 14's ball in text characters, now lit by a lamp. Every character is one pixel: its ray is $(x,\ y,\ -1)$ from the origin, and the ball has radius 3 and centre $(0,\ 0,\ -5)$.

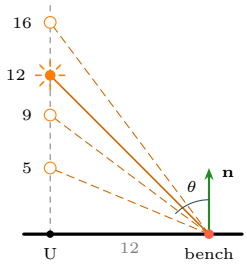
```
W, H = 40, 20                          # characters across and down
ramp = ' .:-=+*#%@'                    # ten levels, from dark to bright
lamp = (-6, 6, 0)                      # the lamp's position: change it
for j in range(H - 1, -1, -1):         # the top row first
    row = ''
    for i in range(W):
        x, y = -1 + 2 * (i + 0.5) / W, -1 + 2 * (j + 0.5) / H
        A = x * x + y * y + 1          # lecture 14: A, and D = B*B - 4*A*C
        D = 100 - 64 * A
        if D < 0:                      # the ray misses the ball
            row += ' '
            continue
        t = (10 - D ** 0.5) / (2 * A)
        px, py, pz = t * x, t * y, -t                  # the hit point
        nx, ny, nz = px / 3, py / 3, (pz + 5) / 3      # the unit normal
        lx, ly, lz = lamp[0] - px, lamp[1] - py, lamp[2] - pz
        r = (lx * lx + ly * ly + lz * lz) ** 0.5       # the distance to the lamp
        c = (nx * lx + ny * ly + nz * lz) / r          # n . l
        b = max(0, c)                                  # the brightness
        row += ramp[int(10 * min(b, 0.999))]
    print(row)
```

3. Run it. Where is the ball brightest, and where is the lamp? Put the lamp at the eye, $(0,\ 0,\ 0)$: why is the middle bright and the rim dim?
4. The Moon. Put the lamp far to the right: at $(60,\ 0,\ -5)$, then at $(60,\ 0,\ -40)$. Which Moon shapes appear?
5. With the lamp at $(-6,\ 6,\ 0)$ again, write `b = c` in place of `b = max(0, c)`. What appears on the dark side? (What does Python do with `ramp[-3]`?)
6. Write `b = 0.15 + 0.85 * max(0, c)`. Which light of the lecture have you added?

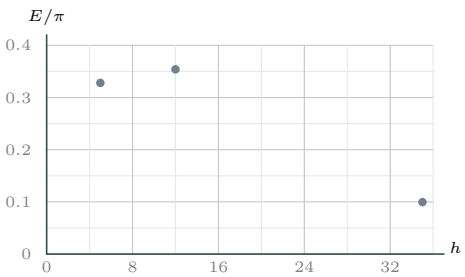
13 Bonus: how high should the lamp hang? ★

about 10 min

For the brave. Distance against tilt: raise the lamp, and its light arrives more steeply, but from farther away.



height h	r	$\cos\theta$	E/π
5	13	0.385	0.328
9			
12	16.97	0.707	0.354
16			
35	37	0.946	0.099



A bench stands on the courtyard's floor, 12 to the right of U. The lamp, $I = 144\pi$, hangs above U on a cable and can be raised or lowered to any height h. It hangs at $h = 12$ now.

(a) Fill the two empty rows of the table: r, then $\cos\theta = h/r$, then $E = I\cos\theta/r^2$ at the bench. Should the lamp go up or down?

(b) Mark your two points in the plot and draw the curve ($E = 0$ at $h = 0$). A very low lamp and a very high lamp both leave the bench dark. Which factor of the lecture is to blame each time?

(c) Show that $E = (I/a^2)\sin^2\theta\cos\theta$, where $a = 12$ is the bench's distance from U. So only the angle counts. Check it with your row for $h = 9$.

(d) Find the best angle and the best height. With derivatives: where is the slope of $\sin^2\theta\cos\theta$ zero? Without: try $\cos\theta = 0.5,\ 0.55,\ 0.6,\ 0.65$. How close was $h = 9$?

(e) A second bench stands 24 from U. Which height suits it best? Can one lamp be best for both?

Answer Key

Check your work *after* attempting everything. If an answer surprises you, that item is worth re-reading in the slides.

1a, b.

distance r	1	2	3	6
tiles lit	1	$2 \times 2 = 4$	$3 \times 3 = 9$	$6 \times 6 = 36$
light on one tile	36	9	4	1

At distance r the beam is r times as wide and r times as high. The same 36 units are shared among r^2 tiles.

1c. Light on one tile $= 36/r^2$. The 36 plays the part of I: it is the light in the beam, and that is the lamp's business alone. The wall's distance only decides over how many tiles the beam is spread.

1d. Sixteen. 2 m is four times 50 cm, so the page gets $1/4^2 = 1/16$ of the light.

1e. $13.6 \times 12.6 \approx 171$ W. At 1 m: $E = 13.6/1^2 = 13.6$ W/m^2. At 2 m: $13.6/4 = 3.4$ W/m^2.

1f. 1360 is 100 times 13.6. A hundred times the light needs a tenth of the distance: 10 cm. Check: $13.6/0.1^2 = 1360$.

2.

stick	rays, of 10	$\mathbf{n} \cdot \mathbf{l}$	$\mathbf{n} \cdot \mathbf{l}'$	rays you predict
1	10	1	$0 + 0.8 = 0.8$	8
2	8	0.8	$-0.36 + 0.64 = 0.28$	about 3
3	6	0.6	$-0.48 + 0.48 = 0$	0
4	0	0	$-0.6 + 0 = -0.6$	none on the side with $\mathbf{n}$

2a. The share of the rays that land on a stick is its $\mathbf{n} \cdot \mathbf{l}$: 10, 8, 6 and 0 of 10. The four sticks are equally long, and every ray carries the same light. So the count is the light per unit of stick: E, up to a constant.

2b. Stick 3 runs along (0.6, 0.8), the direction of the light itself: it is edge on, and the light slides along it. Stick 4's minus sign: the light now arrives on the side that $\mathbf{n}$ points away from. For the side with the normal, the lamp is behind, and that side gets nothing.

2c. $(-0.6,\ 0.8) \cdot (0,\ 20) = 16$: twenty times the right 0.8, because the arrow is 20 long. A cosine is never above 1. The dot product is the cosine only when both arrows have length 1.

3a. 5 of 152: about 3.3% nearer, so H should be about 6.6% larger. The table: $1410/1320 = 1.068$. (The roof on the lecture's slide was 300 m nearer: 0.000 000 2 %.)

3b. June: $1320 \times 0.99 \approx 1307$ W/m^2. December: $1410 \times 0.57 \approx 804$ W/m^2.

3c. The tilt. From June to December the cosine falls from 0.99 to 0.57, a loss of 42%. The distance works the other way, and adds only 7%.

3d. Straight at the sun: $\mathbf{n} = \mathbf{l}$, the panel tilted 55° from level, toward the south. Then $\cos\theta = 1$ and it collects the full $H = 1410$ W/m^2: more than flat ground gets at noon in June.

4a. U gets the most and J the least: each step along the floor is farther from the lamp and more tilted.

		$\mathbf{p} - \mathbf{x}$	r	$\mathbf{l}$	$\mathbf{n} \cdot \mathbf{l}$	E
1	U, floor	(0, 12, 0)	12	(0, 1, 0)	1	$144\pi/144 = \pi$
2	A, ball	(0, 7.2, 9.6)	12	(0, 0.6, 0.8)	$0.36 + 0.64 = 1$	π
3	D, ball	(0, 10.8, 14.4)	18	(0, 0.6, 0.8)	$-0.36 - 0.64 = -1$	0
4	J, floor	(0, 12, 16)	20	(0, 0.6, 0.8)	0.6	$144\pi \times 0.6/400 = 0.216\,\pi$
5	J, wall	(0, 12, 16)	20	(0, 0.6, 0.8)	0.8	$144\pi \times 0.8/400 = 0.288\,\pi$
6	K, wall	(−15, 12, 16)	25	(−0.6, 0.48, 0.64)	0.64	$144\pi \times 0.64/625 \approx 0.147\,\pi$

4b. Both are 12 from the lamp, and both face it exactly: $\mathbf{n} = \mathbf{l}$. Twelve away and facing, this lamp gives $E = \pi$.

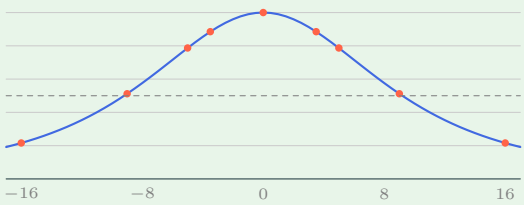
4c. Through A, through the ball's centre and through D. The lamp, A, D and J lie on one line, so the arrow toward the lamp is the same at all three points.

4d. The wall: $\mathbf{n} \cdot \mathbf{l} = 0.8$ against the floor's 0.6. The light meets the wall more squarely. Two walls with the same paint are turned differently to the lamp, so they get different E and look differently bright. The corner shows as a jump in brightness.

4e. No. The ball stands between J and the lamp, so J lies in the ball's shadow. The formula uses only the point, its normal and the lamp's position. It never asks whether something is in the way. That needs a ray toward the lamp: next lecture.

4f. $0.36 + 0.2304 + 0.4096 = 1$.

5a.



E/π is 1 at U, 0.512 at G and 0.216 at J, and the same at depths -9 and -16. The curve is a hill with its top under the lamp: a pool of light on the floor.

5b. At G the distance costs more: $(12/15)^2 = 0.64$ against 0.8 for the tilt. At J: $(12/20)^2 = 0.36$ and 0.6.

5c. $r = 12/\cos\theta$, so $I/r^2 = \pi\cos^2\theta$; times the tilt, $E = \pi\cos^3\theta$. At depth 3.5: $0.96^3 = 0.885$.

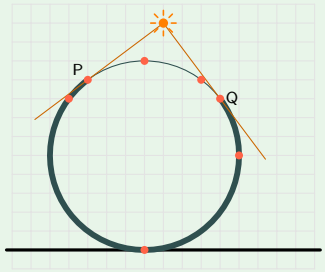
5d. At about 9 (more exactly 9.2): three quarters of the lamp's height. G, at depth 9, has 0.512.

5e. Each lamp is 15 away and gives $0.512\,\pi$: together $1.024\,\pi$, about the π under one lamp alone. (With both on, U gets $\pi + 0.17\,\pi$.) Lamps one and a half heights apart light the ground between them almost evenly.

6a.

	$\mathbf{p}-\mathbf{x}$	$\mathbf{u}\cdot(\mathbf{p}-\mathbf{x})$	the lamp is
S	$(-2,\ 3)$	$-6+12=6$	in front
P	$(4,\ 3)$	$-12+12=0$	grazing
Q	$(-3,\ 4)$	$-12+12=0$	grazing
M	$(-4,\ 7)$	$-20+0=-20$	behind
V	$(5,\ 4)$	$-20+12=-8$	behind
F	$(1,\ 12)$	$0-60=-60$	behind

6b, c.



The two lines only touch the ball: they are tangents. At P and Q the line to the lamp is perpendicular to $\mathbf{u}$, which is what $\mathbf{u}\cdot(\mathbf{p}-\mathbf{x}) = 0$ says. Each tangent is 5 long.

The lamp lights only the arc from P over T to Q. The thick line, all the rest, is far more than half. In fact $\mathbf{u}$ at P and $\mathbf{u}$ at Q are perpendicular, $(-3,\ 4)\cdot(4,\ 3) = 0$, so the lit arc is a quarter of the outline.

6d. $\mathbf{u}\cdot(1,\ 7)$ is 35 at T, 31 at S, 25 at P and at Q, 5 at M, 17 at V, and -35 at F. Six points are lit; only F is dark. The border is the diameter perpendicular to $(1,\ 7)$: exactly half of the ball is lit. The sun lights half of any ball. A nearby lamp lights less than half, and the nearer it is, the less.

6e. About $-I/145$: negative light. Added to the second lamp's share it would take light away, and the pixel would be darker than with the first lamp switched off. The max turns it into 0.

7a, b.

eye	$\cos\theta$	looks ... wide	light toward it
B	0.8	4	8
C	0.6	3	6
D	0	0: the patch is edge on	0

Each width is $5\cos\theta$, and the light toward a direction is $L_r \times \cos\theta$. The patch is equally bright from everywhere, but from the side it looks narrower, so less light goes that way. It is Exercise 2's count with the arrows turned round.

7c. 0.5, and $\cos 60° = 0.5$. The cosine of a direction is its height on the dome: the radius is 1, and the height is the side of the right triangle that lies along "up".

7d. Cosines 0.7, 0.5, 0.3, 0.1; products $0.28\,\pi$, $0.2\,\pi$, $0.12\,\pi$, $0.04\,\pi$. With the cap's $0.36\,\pi$ the sum is $1.0\,\pi$: exactly π.

7e. $\pi/2\pi = 1/2$: half of the dome's area. Two bands: each has area $2\pi \times 0.5 = \pi$, and their middle heights are 0.25 and 0.75, so the sum is $0.25\,\pi + 0.75\,\pi = \pi$ again. The number of bands does not matter: their middle heights are spread evenly from 0 to 1, so their mean is always 1/2.

7f. $\pi\,L_r = 188$, so $L_r = 188/\pi \approx 60$: the worked row of Exercise 8.

8a.

surface	sent back	absorbed	L_r	k
fresh snow	283	31	90	0.29
brick	94	220	30	0.095
tarmac	31	283	10	0.032

row	E	k	L_r
1	628	same	120
2	78.5	same	15
3	157	same	30
4	same	same	same
5	same	0.095	30
6	0	same	0
7	628	0.095	60

$L_r = (R/\pi) \times 100\pi = 100\,R$, and $k = R/\pi$.

8b. Rows 1, 2, 3, 6 and 7 moved E: more lamps, the distance, the tilt. Rows 5 and 7 moved k: the paint. Row 4 moved nothing: a Lambertian surface looks the same from everywhere. E belongs to the lighting. k belongs to the material.

8c. A dark wall in strong light and a pale wall in weak light. Only the product $k\,E$ reaches the camera.

8d. 314 in every row: the light that arrives is either sent back or absorbed. With $R = 1.2$ the surface would send back more light than it receives. It would be making light.

8e. $1/\pi \approx 0.32$, for $R = 1$. Fresh snow, with 0.29, is close to it.

8f. Its R. R is a share of the light. k is not: it is the number that turns the light arriving, E, into the light that one direction gets, L_r. One eye catches only a tiny part of the light that is sent back.

9a.

pixel	division	surface	s
1	$0.4/0.8 = 0.35/0.7 = 0.3/0.6$	floor	0.5
2	$0.24/0.2 = 0.48/0.4 = 0.96/0.8$	ball	1.2
3	$0.3/0.6 = 0.15/0.3 = 0.1/0.2$	wall	0.5
4	$0.04/0.2 = 0.08/0.4 = 0.16/0.8$	ball	0.2
5	$1.2/0.8 = 1.05/0.7 = 0.9/0.6$	floor	1.5
6	by the floor: 0.31, 0.39, 0.58	none	—

9b. The light that arrives. Pixel 4's hit gets far less of it: the surface there is turned away from the lamp. (Its $s = 0.2$ is Exercise 12's ambient share alone.)

9c. 1.2 and 1.05 become 0.999, so the bytes are (255, 255, 230): almost white. The sandstone's 8 : 7 : 6 is lost, because two channels were cut off. The floor there is burnt out.

9d. $(0.4 + 0.1)/2 = 0.25$, $(0.35 + 0.2)/2 = 0.275$, $(0.3 + 0.4)/2 = 0.35$. Lecture 15: half of the pixel's samples saw the ball and half the floor.

9e. $L_r = (R/\pi)\,E$, channel by channel: floor (0.8, 0.35, 0), wall (0.6, 0.15, 0), ball (0.2, 0.2, 0). The ball is a dark, dull yellow. No blue light arrives, so none can come back, and its R is small in red and green. Under orange street lamps a blue car looks almost black.

10a. `scene.hit` once. `illuminate` three times: the loop calls it once for each light in the list. `evaluate` twice: each lamp's `illuminate` calls it, in the line `k = rec.surface.material.evaluate(l, v, n)`. The ambient light never calls it. It reads the material's `albedo` in its place.

10b. The light that **x**, the hit point, sends to the eye because of this one light. `trace` adds the three shares, starting from black, and returns the sum: the pixel's colour.

11a. P: **p**, r, $\mathbf{n} \cdot \mathbf{l}$, E, I. M: R, k. A: I_a. T: the number of lights, the background colour. H: the surface that was hit, and **n**.

11b. `E = max(0, n.dot(self.l)) * self.H`; the material is asked with `self.l`; `return k * E`.

11c. The lines for **x**, r and **l**. A directional source has no position, so there is no distance to work out, and its **l** is stored, the same for every hit. Nothing else changes: the new light is one more entry in the scene's list. The materials and `trace` stay as they are.

11d. The cosine is about how much light arrives, and that is the same for every material. The material answers only what it does with the light.

12a. $\mathbf{x} = \mathbf{e} + 15\,\mathbf{d} = (0,\ 4.8,\ -9.6)$, the point A. $\mathbf{p} - \mathbf{x} = (0,\ 7.2,\ 9.6)$, $r = 12$, $\mathbf{l} = (0,\ 0.6,\ 0.8)$. $\mathbf{v} = (0,\ 0,\ 1)$. $E = 1 \times 144\pi/144 = \pi$. $k = (0.2,\ 0.4,\ 0.8)/\pi$. It returns $k\,E = (0.2,\ 0.4,\ 0.8)$, the ball's own reflectance. The ball never uses **v**: its `evaluate` gets **l**, **v** and **n** and ignores all three.

12b.

	ambient	lamp	$\mathbf{n} \cdot \mathbf{l}_s$	sun	s
the ball at A	0.2	1	$0.36 - 0.64 = -0.28$	0	1.2
the wall at J	0.2	0.288	-0.8	0	0.488
the floor at J	0.2	0.216	0.6	0.3	0.716

The sky: the background colour, (0.5, 0.7, 1), from the `else` branch.

12c. A: $1.2 \times (0.2,\ 0.4,\ 0.8) = (0.24,\ 0.48,\ 0.96)$, pixel 2. U: $1.5 \times (0.8,\ 0.7,\ 0.6) = (1.2,\ 1.05,\ 0.9)$, pixel 5.

12d. The sun's share becomes $0.5 \times (-0.8) = -0.4$, and the sum $0.2 + 0.288 - 0.4 = 0.088$ instead of 0.488. The wall is darker with the sun than without it. Every surface that faces away from one light would lose some of what the other lights give: dark patches where there should be light.

12e. Neither the lamp's 0.216 (the ball hides the lamp) nor the sun's 0.3 (the wall hides the sun). Only the ambient share is right: $s = 0.2$, not 0.716. The program draws that piece of floor more than three times too bright. It lacks the question "is something between the point and the light?": lecture 11's shadow ray, now in 3D. Next lecture.

T1. Both make it darker. The tilt should halve the light ($\cos 60° = 0.5$). Twice the distance should leave a quarter: the distance costs more.

T2. The paper looks equally bright from every side: it is close to Lambertian. The glossy cover shows a bright spot,

the reflection of the torch, and the spot moves when your head moves. That is the shiny top layer: lecture 17.

T3. The upper left is brightest: the lamp is above the eye and to its left. With the lamp at the eye, the middle of the ball faces the lamp, $\mathbf{n} \cdot \mathbf{l} = 1$, and the rim is turned almost edge on to it.

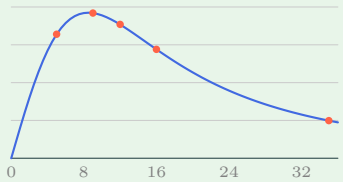
T4. A half moon, then a thin crescent. (With the lamp behind the ball, at $(0,\ 0,\ -60)$, the side that we see faces away from it: a new moon.)

T5. Bright characters, `@` and `%`, on the side that should be dark. A negative index counts from the end of the string, so `ramp[-1]` is the brightest character. The lecture's "never negative light" slide: leave out the max, and wrong light appears.

T6. The dark side is no longer empty: it is drawn with dots, and the ball looks whole. That is the ambient light, a constant share at every hit.

13a. $h = 9$: $r = 15$, $\cos\theta = 0.6$, $E = 144\pi \times 0.6/225 = 0.384\,\pi$. $h = 16$: $r = 20$, $\cos\theta = 0.8$, $E = 144\pi \times 0.8/400 = 0.288\,\pi$. The lamp should go down: $0.384\,\pi$ at $h = 9$ against $0.354\,\pi$ at $h = 12$.

13b.



The curve climbs steeply, peaks between $h = 8$ and 9, and falls slowly. A very low lamp is near, but its light only grazes the floor at the bench: the tilt, $\cos\theta$, is to blame. A very high lamp shines almost straight down on the bench, but from far away: the inverse square, $1/r^2$.

13c. In the triangle of lamp, U and bench, $\sin\theta = a/r$, so $1/r^2 = \sin^2\theta/a^2$, and $E = I\cos\theta/r^2 = (I/a^2)\sin^2\theta\cos\theta$. Here $I/a^2 = \pi$. For $h = 9$: $0.8^2 \times 0.6 = 0.384$.

13d. The slope is $2\sin\theta\cos^2\theta - \sin^3\theta$, and it is zero when $\tan^2\theta = 2$: $\cos\theta = 1/\sqrt{3} \approx 0.577$, $\theta \approx 55°$. Then $h = a/\tan\theta = 12/\sqrt{2} \approx 8.5$ and $E = 0.385\,\pi$. By trying: 0.375, 0.384, 0.384, 0.375, so the top lies between 0.55 and 0.6. The lamp at $h = 9$ was within half a percent of the best.

13e. $24/\sqrt{2} \approx 17$: twice as high. No: every point of the floor wants its own height, $a/\sqrt{2}$. One lamp's height is a compromise, and large places are lit by several lamps.