Extra 3: Colour and Wavelength

Light Is a Spectrum

CS 453 — Computer Graphics

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Forman Christian University

Today: from three numbers to a spectrum

Extra 3: self-study, not on the exams



$(0.70,\ 0.10,\ 0.08)$

lectures 15 to 20 and Extra 2:
a colour is three numbers

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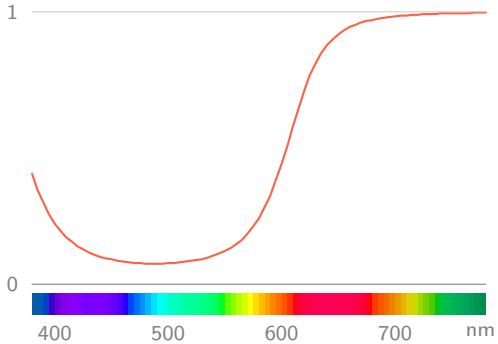
lectures 15 to 20 and Extra 2:
a colour is three numbers

the same red paint, at every wavelength



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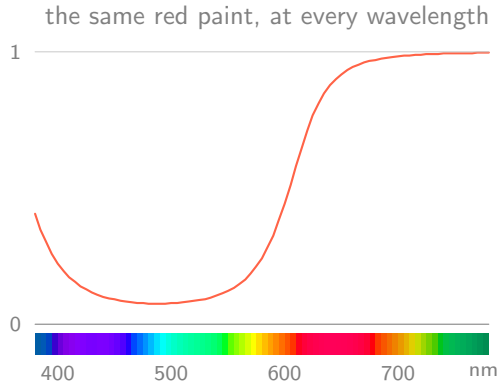

- Light is a curve over wavelength; three numbers are a shadow of it.

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lectures 15 to 20 and Extra 2:
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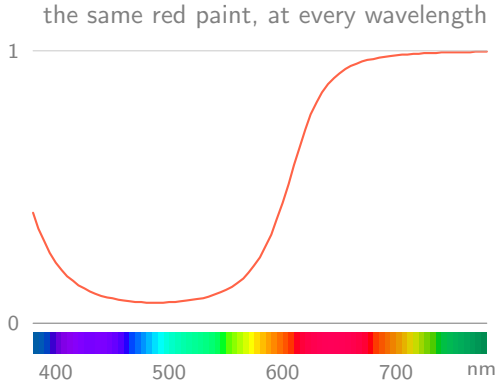


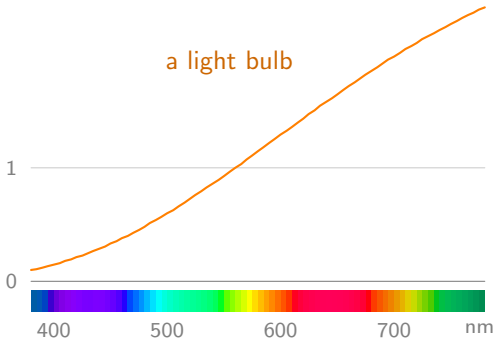
- Light is a curve over wavelength; three numbers are a shadow of it.
- Trace wavelengths, and prisms, rainbows and soap bubbles come for free.

Colour is a projection

A spectrum in, three numbers out

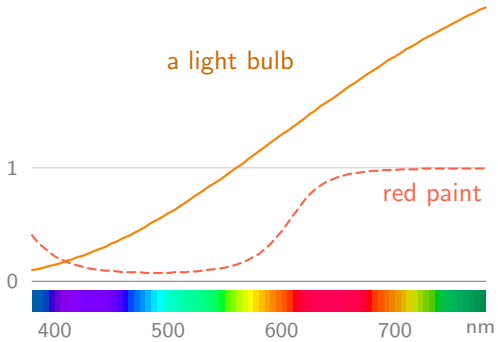
Light is power at each wavelength

- A lamp: how much power at each wavelength.



Light is power at each wavelength

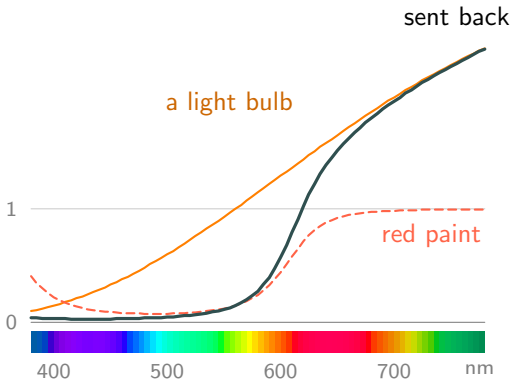
- A lamp: how much power at each wavelength.
- A paint: what share it sends back, at each wavelength.



Light is power at each wavelength

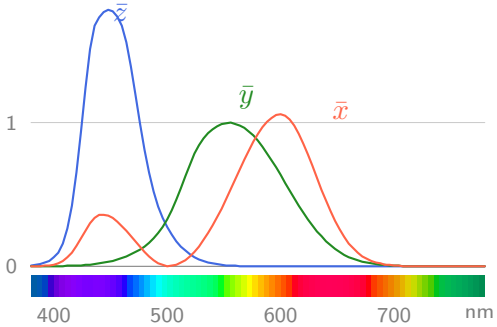
- A lamp: how much power at each wavelength.
- A paint: what share it sends back, at each wavelength.
- What leaves: the two multiplied, wavelength by wavelength.

$$L_{\text{out}}(\lambda) = s(\lambda)\, L_{\text{in}}(\lambda)$$



The eye keeps three numbers

CIE 1931: the matching functions



The eye keeps three numbers

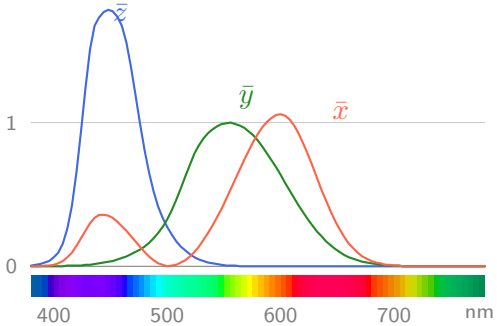
$$X = \int L(\lambda)\,\bar{x}(\lambda)\,d\lambda$$

$$Y = \int L(\lambda)\,\bar{y}(\lambda)\,d\lambda$$

$$Z = \int L(\lambda)\,\bar{z}(\lambda)\,d\lambda$$

- Three weighted sums of the spectrum.

CIE 1931: the matching functions



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- Three weighted sums of the spectrum.
- Y is how bright it looks.

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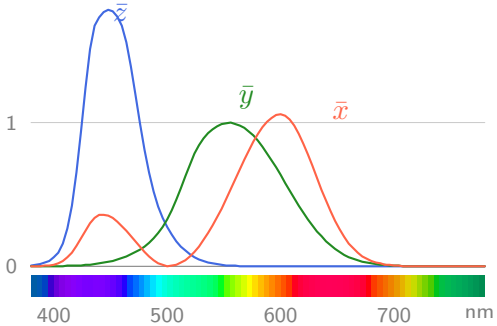

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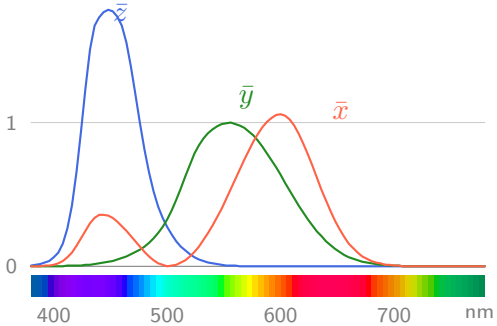
- Three weighted sums of the spectrum.
- Y is how bright it looks.
- Equal X, Y, Z: the same colour to the eye.

CIE 1931: the matching functions



From X, Y, Z to the screen's red, green and blue

$$\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 3.241 & -1.537 & -0.499 \\ -0.969 & 1.876 & 0.042 \\ 0.056 & -0.204 & 1.057 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

sRGB; then lecture 19's gamma

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sRGB; then lecture 19's gamma

Try first: light of 550 nm only.

$(X, Y, Z) = (0.434,\ 0.994,\ 0.009)$



the closest the screen can show

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- One wavelength is too pure for the screen: red and blue would have to be below 0.

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sRGB; then lecture 19's gamma

- One wavelength is too pure for the screen: red and blue would have to be below 0.
- Keep X, Y, Z until the very end; clip once, when the pixel is written.

Try first: light of 550 nm only.

$(X, Y, Z) = (0.434,\ 0.994,\ 0.009)$

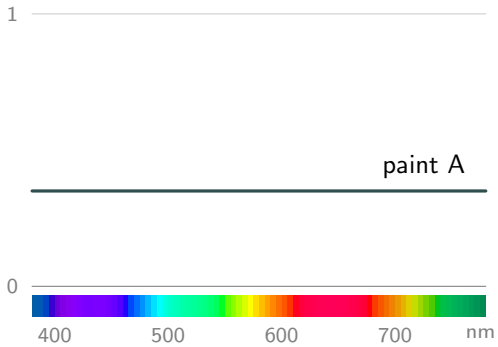
$(r, g, b) = (-0.13,\ 1.45,\ -0.17)$



the closest the screen can show

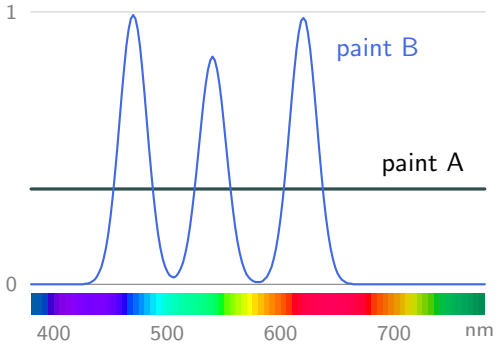
Two paints, the same three numbers

- Paint A: 0.35 at every wavelength, a grey.



Two paints, the same three numbers

- Paint A: 0.35 at every wavelength, a grey.
- Paint B: three narrow bands.



Two paints, the same three numbers

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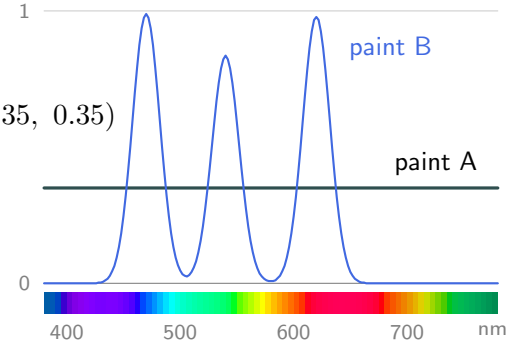
under an even white lamp:



A B

both $(0.35,\ 0.35,\ 0.35)$



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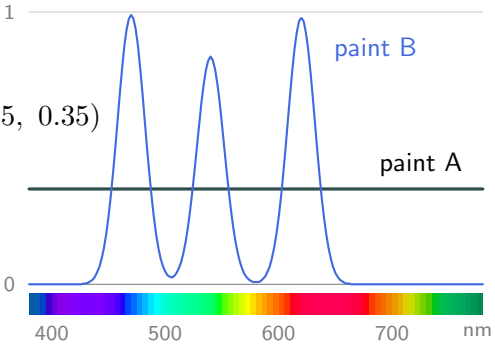
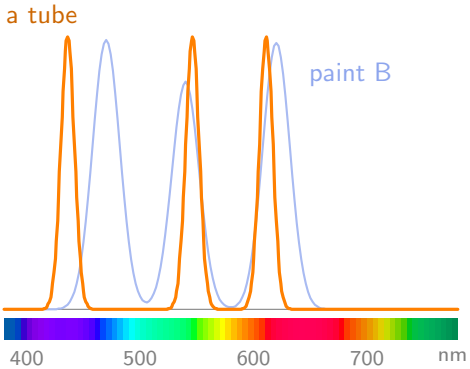


- Different spectra, the same X, Y, Z: **metamers**. The eye cannot tell them apart.

Change the lamp, and the paints part

under the tube (white-balanced to it):

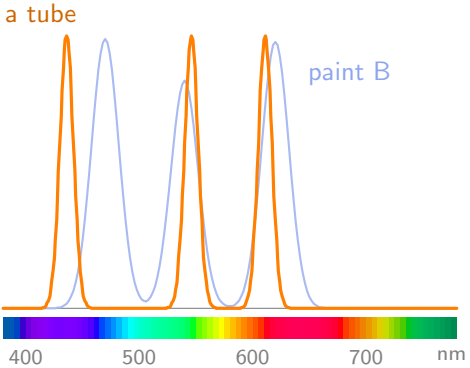


Change the lamp, and the paints part

under the tube (white-balanced to it):



spectral



- Spectral: A stays grey; B turns yellow-green.

Change the lamp, and the paints part

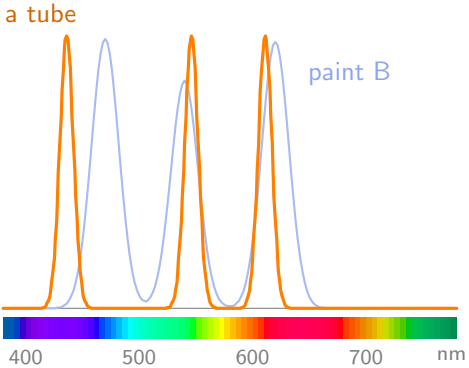
under the tube (white-balanced to it):



spectral



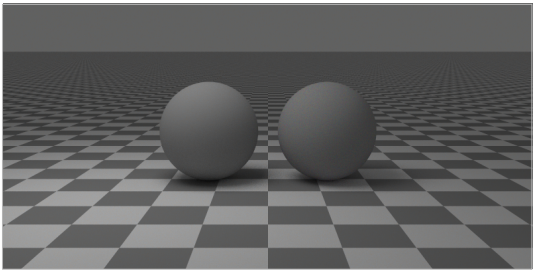
red, green, blue



- Spectral: A stays grey; B turns yellow-green.
- Red, green, blue: both are $(0.35, 0.35, 0.35)$; they never part.

Rendered: two paints, two lamps

paint A on the left, paint B on the right; traced with spectra



an even white lamp

Rendered: two paints, two lamps

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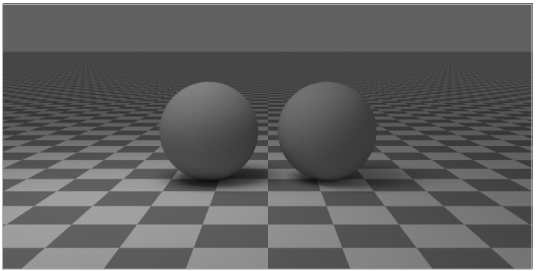

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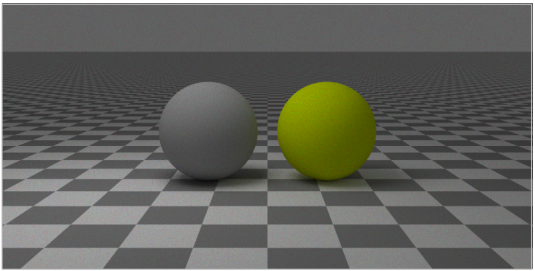

the fluorescent tube

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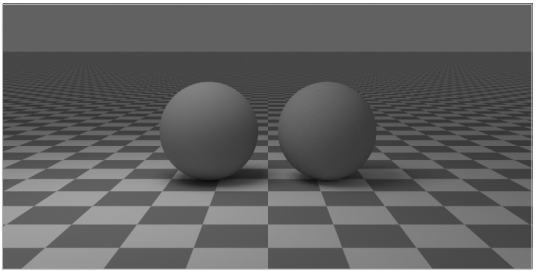

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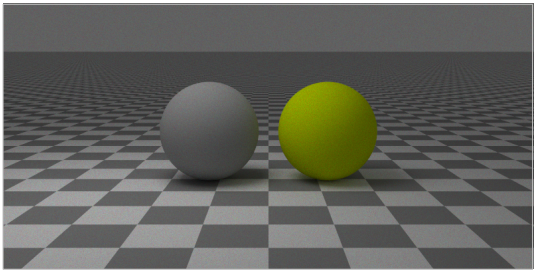
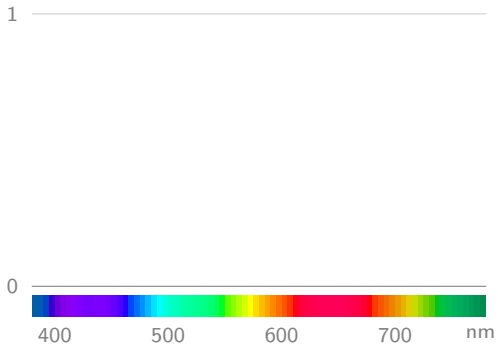


the fluorescent tube

- The same two balls: identical under one lamp, far apart under the other.
- Only a renderer that keeps the spectrum can draw the right picture.

From an artist's colour to a spectrum

- The artist gives (r, g, b); the tracer needs a curve.



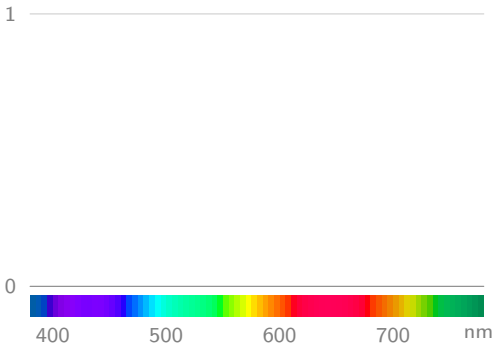
From an artist's colour to a spectrum

- The artist gives (r, g, b); the tracer needs a curve.

$$s(\lambda) = S(c_0 t^2 + c_1 t + c_2), \quad t = \frac{\lambda - 380}{400}$$

$$S(x) = \frac{1}{2} + \frac{x}{2\sqrt{1 + x^2}}$$

S: an S-shaped curve from 0 to 1



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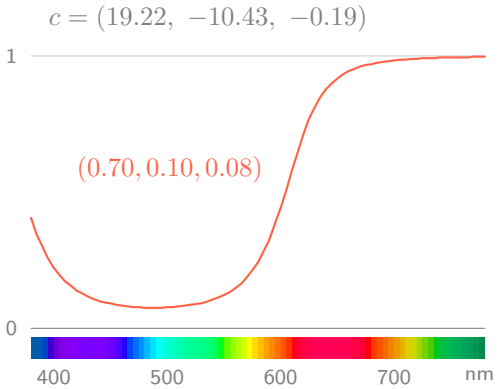
S: an S-shaped curve from 0 to 1

- Three numbers c per colour, found once.



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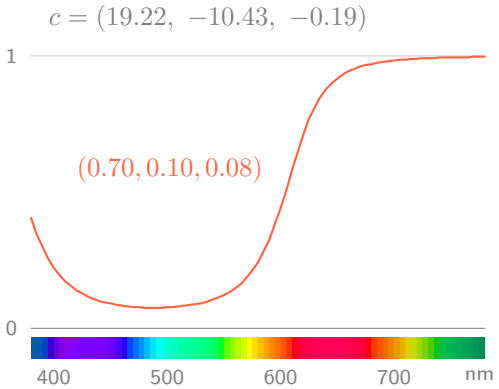


- Three numbers c per colour, found once.
- Smooth, between 0 and 1, the right colour.

Recap: colour is a projection

light: power at each wavelength; bounces multiply

the eye: $X, Y, Z \rightarrow$ the screen: r, g, b by one matrix

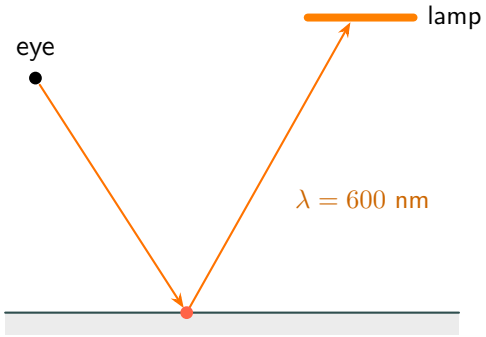
metamers part under another lamp; three numbers cannot see it

Tracing wavelengths

Lecture 20's Monte Carlo, over one more variable

One wavelength per path

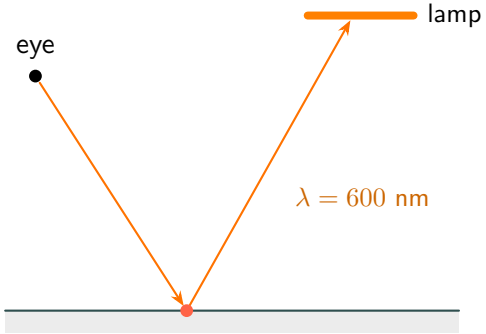
- Pick λ evenly in 380 to 780 nm: $p(\lambda) = 1/400$.



every albedo and glow:
one number, read at λ

One wavelength per path

- Pick λ evenly in 380 to 780 nm: $p(\lambda) = 1/400$.
- Trace lecture 20's path at that λ: it brings back one number, $L(\lambda)$.



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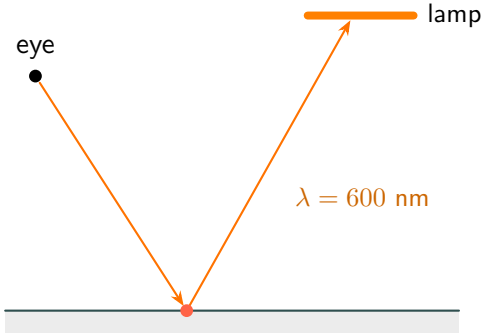
- Trace lecture 20's path at that λ: it brings back one number, $L(\lambda)$.
- The pixel adds:

$$\frac{1}{106.9} \cdot \frac{L(\lambda)\left(\bar{x}(\lambda),\ \bar{y}(\lambda),\ \bar{z}(\lambda)\right)}{p(\lambda)}$$



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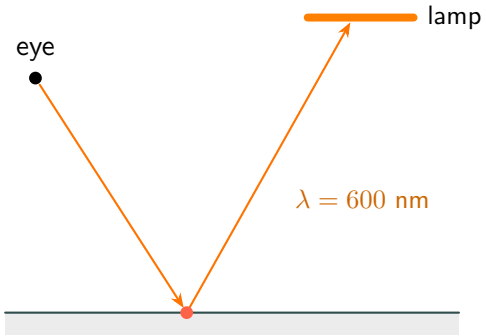
$$\frac{1}{106.9} \cdot \frac{L(\lambda)\left(\bar{x}(\lambda),\ \bar{y}(\lambda),\ \bar{z}(\lambda)\right)}{p(\lambda)}$$

- Averaged over many paths: X, Y, Z, the integrals of the last part.



every albedo and glow:
one number, read at λ

Worked: one path at 600 nm

Try first: a path at 600 nm brings back $L = 0.5$. What does it add?

$(\bar{x}, \bar{y}, \bar{z})(600) = (1.056,\ 0.634,\ 0.000); \quad \dfrac{1}{106.9} \cdot \dfrac{1}{1/400} = 3.74$

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$$(X, Y, Z) = 3.74 \times 0.5 \times (1.056,\ 0.634,\ 0.000) = (1.98,\ 1.19,\ 0.00)$$

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$(r, g, b) = (4.58,\ 0.31,\ -0.13)$: a bright orange-red



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$(r, g, b) = (4.58,\ 0.31,\ -0.13)$: a bright orange-red





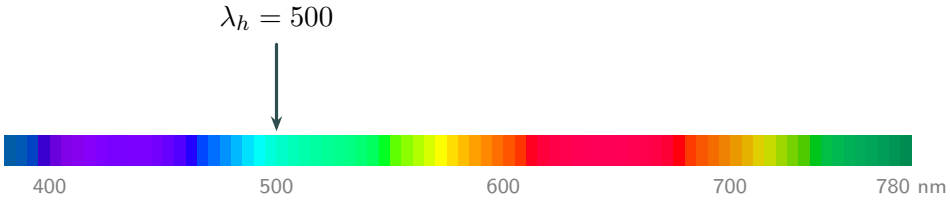
$\longrightarrow$ the average: the right colour



- Each path is one pure hue, far too bright; only the average is right.

Hero wavelengths: four per path

$\xi = 0.3$: $\lambda_h = 380 + 400 \times 0.3 = 500$ nm



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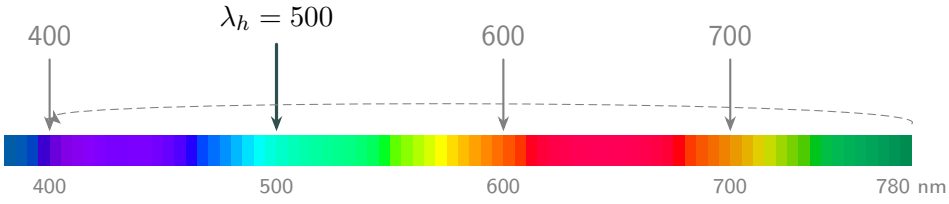
$500 + 300 = 800$: wrapped round to 400



- Carry three more, a quarter of the range apart, wrapped round.

$\lambda_j = 380 + (\lambda_h - 380 + 100\,j) \bmod 400, \quad j = 0, 1, 2, 3$

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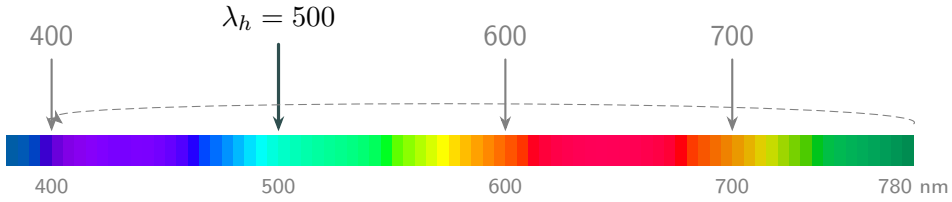
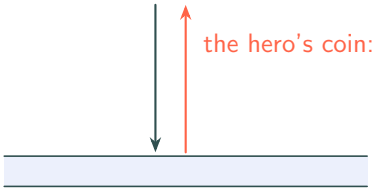


- Carry three more, a quarter of the range apart, wrapped round.

  $\lambda_j = 380 + (\lambda_h - 380 + 100\,j) \bmod 400, \quad j = 0, 1, 2, 3$

- Where nothing depends on λ: four wavelengths for the price of one path.

When a coin depends on the wavelength

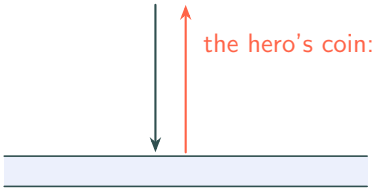


chance R of reflecting:
a different R at each wavelength

Try first: each wavelength's $r_j = R_j/R_h$.

λ	500 (hero)	600	700	400
R	0.071	0.058	0.015	0.000

When a coin depends on the wavelength



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a different R at each wavelength

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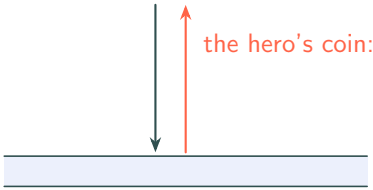

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each adds $\dfrac{w_j\, L(\lambda_j)}{r_0 + r_1 + r_2 + r_3}$

here the sum is 2.03

- w_j: each wavelength's own share, R_j/R_h here. The sum: Extra 1's balance heuristic.

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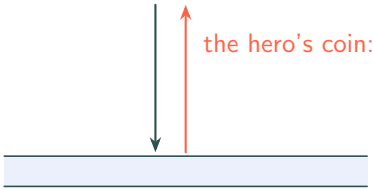

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- w_j: each wavelength's own share, R_j / R_h here. The sum: Extra 1's balance heuristic.
- Glass that bends each λ its own way: the others' r is 0, and the hero counts as four.

One wavelength, in Python

```
class Spectrum:
    def __init__(self, values):   # at 380, 381, ..., 780 nm
        self.values = values
    def at(self, lam):
        return self.values[round(lam) - 380]
```



One wavelength, in Python

```
class Spectrum:
    def __init__(self, values):   # at 380, 381, ..., 780 nm
        self.values = values
    def at(self, lam):
        return self.values[round(lam) - 380]
```




```
def trace(ray, depth, lam):
    ...
    if isinstance(material, Lambertian):
        return material.albedo.at(lam) * trace(Ray(x, n + u), depth - 1, lam)
    elif isinstance(material, DiffuseLight):
        return material.emission.at(lam)
```

One wavelength, in Python

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class Spectrum:
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```




```
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        return material.albedo.at(lam) * trace(Ray(x, n + u), depth - 1, lam)
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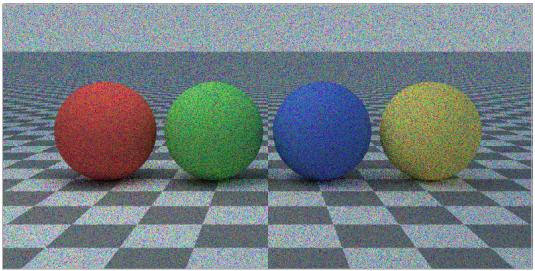
        return material.emission.at(lam)

lam = 380 + 400 * random.random()
L = trace(ray, max_depth, lam)
xyz = xyz + L * cmf(lam) * 400 / 106.9
```

- `cmf`: the three curves at `lam`, as a `Vec3`.

Rendered: one wavelength against four

16 paths per pixel in both; the same scene, the same number of rays



one wavelength per path

Rendered: one wavelength against four

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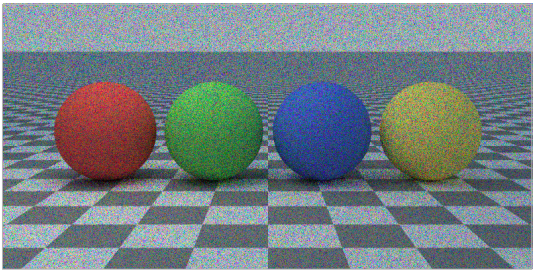
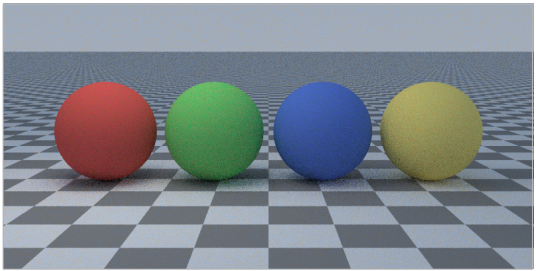


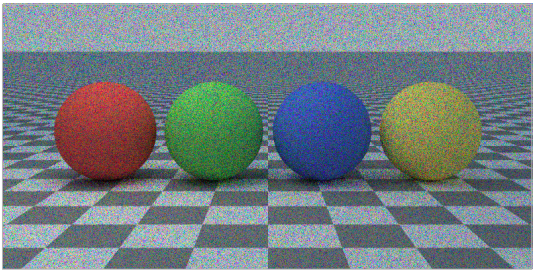
one wavelength per path



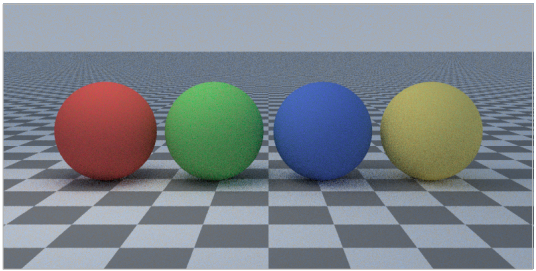
four: a hero and three more

Rendered: one wavelength against four

16 paths per pixel in both; the same scene, the same number of rays



one wavelength per path



four: a hero and three more

- One wavelength: specks of every colour. Four: almost only the noise of lecture 20.

Quick check 1: spectra and paths

One path adds $3.74\,L(\lambda)\,(\bar{x}, \bar{y}, \bar{z})(\lambda)$; heroes: λ_h and three more, 100 nm apart.

1. Two spectra give the same X, Y, Z. What are they called?
2. At 450 nm, $L = 0.2$ ($\bar{x} = 0.344$, $\bar{z} = 1.782$): X and Z?
3. The hero is 650 nm. Which three wavelengths go with it?
4. All four r_j are 1. What is each term divided by?

- A path is lecture 20's, read at its wavelengths; the pixel adds up X, Y, Z.

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2.	At 450 nm, $L = 0.2$ ($\bar{x} = 0.344$, $\bar{z} = 1.782$): X and Z?	0.26, 1.33
3.	The hero is 650 nm. Which three wavelengths go with it?	750, 450, 550
4.	All four r_j are 1. What is each term divided by?	4

- A path is lecture 20's, read at its wavelengths; the pixel adds up X, Y, Z.

Recap: tracing wavelengths

one path, one λ: adds $L(\lambda)\,(\bar{x}, \bar{y}, \bar{z})(\lambda) \,/\, p(\lambda)$

hero: four λ's per path, against colour noise

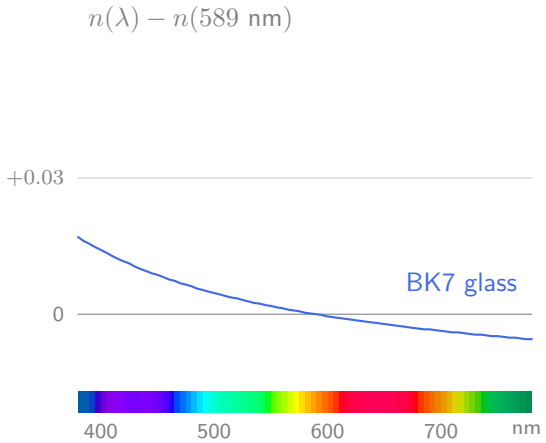
a choice that depends on λ: divide by $r_0 + r_1 + r_2 + r_3$

Bending by colour

Lecture 18's glass, one wavelength at a time

The index depends on the wavelength

- Blue bends more: n is larger at short λ.



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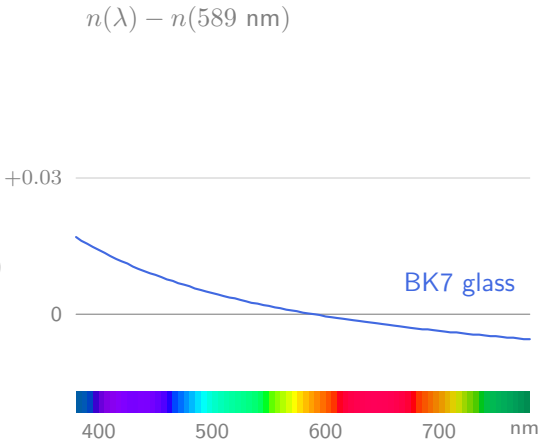
Cauchy: $n(\lambda) = A + \dfrac{B}{\lambda^2}$

BK7: $A = 1.5046,\ \ B = 4200$ nm^2

1.531 at 400 nm, 1.518 at 550, 1.513 at 700



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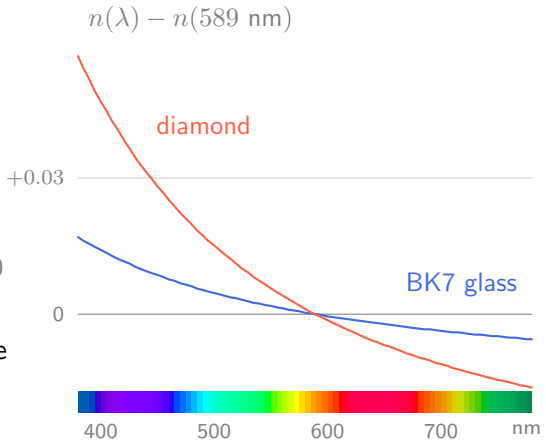
- Diamond: 2.464 to 2.406, three times the spread.

  exact: Sellmeier's formula, from the maker



Worked: lecture 18's Snell at two wavelengths

Try first: 60° from air into BK7, at 450 and 650 nm.

$n(450) = 1.5253,\ \ n(650) = 1.5145;\ \ \sin\theta_t = \sin 60°/n$

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450 nm: $\sin\theta_t = 0.866/1.5253 = 0.5678,\ \ \theta_t = 34.60°$

650 nm: $\sin\theta_t = 0.866/1.5145 = 0.5718,\ \ \theta_t = 34.88°$

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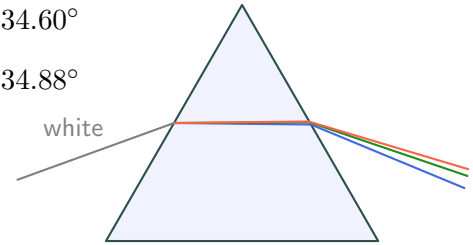
- A prism bends twice, and its faces are tilted: the colours leave $1.6°$ apart.



a prism (spread drawn 6 times wider)

Lecture 18's glass, with a wavelength

```
class Glass(Material):
    def __init__(self, a, b):   # Cauchy:  BK7 is 1.5046, 4200
        self.a, self.b = a, b
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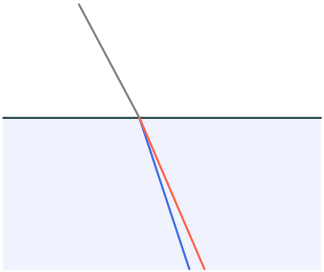

one path, one λ, one ray

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def shade_glass(ray, rec, depth, lam):
    ...  x, d, n as in lecture 18
    ng = rec.surface.material.index(lam)
    ni, nt = (1.0, ng) if rec.front_face else (ng, 1.0)
    t = refract(d, n, ni, nt)
    R = schlick(-d.dot(n), ni, nt)
    ...  trace(Ray(x, t), depth - 1, lam)
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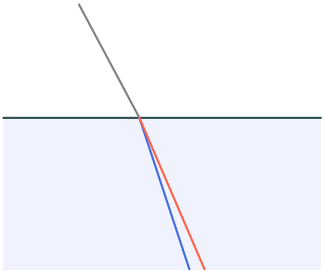

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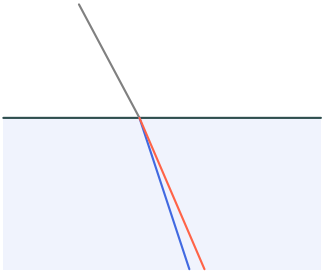

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- One path per wavelength: nothing else changes.

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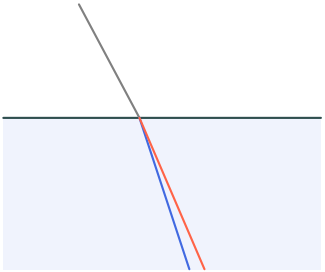
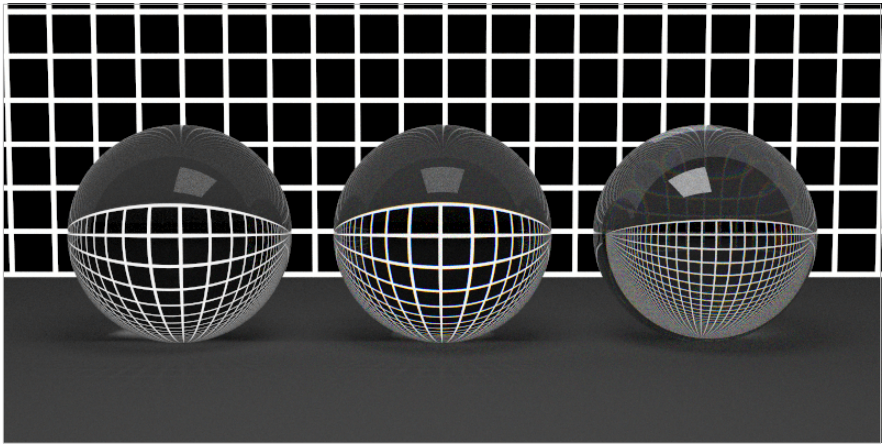


one path, one λ, one ray

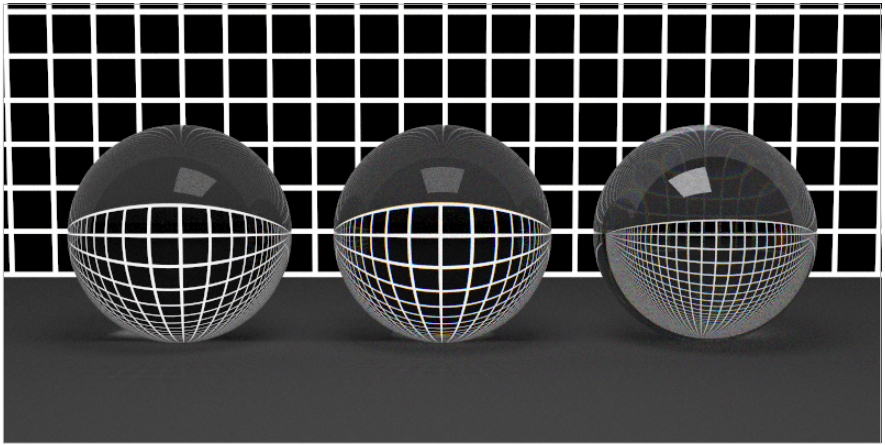
- One path per wavelength: nothing else changes.
- Hero wavelengths: the bent ray follows the hero; the others drop out.

Rendered: three glass balls, three indices



1.5185 at every λ BK7 diamond

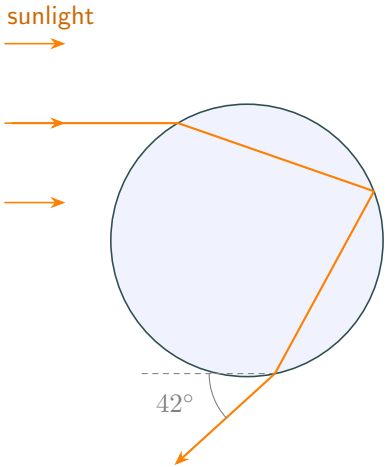
Rendered: three glass balls, three indices



1.5185 at every λ BK7 diamond

- Seen through the glass, white lines grow coloured edges; diamond's are widest.

The rainbow: one bounce inside a drop



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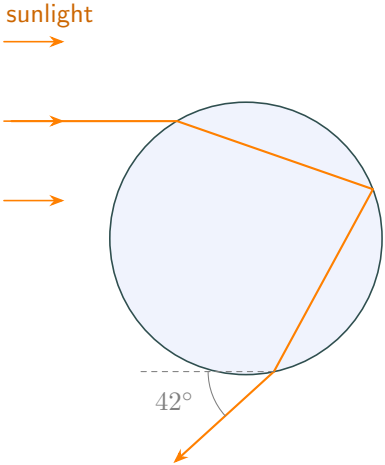

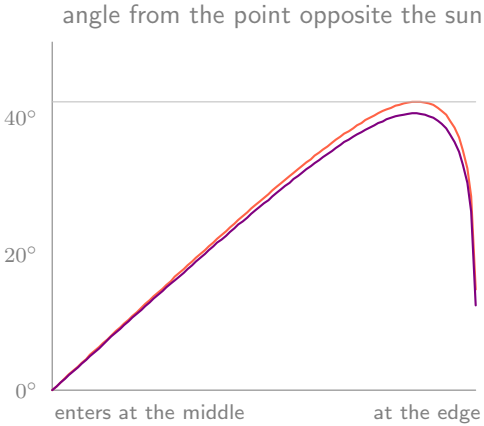

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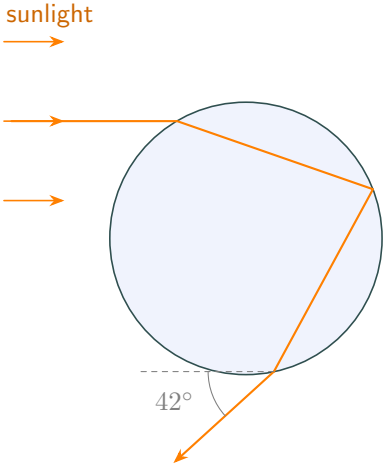
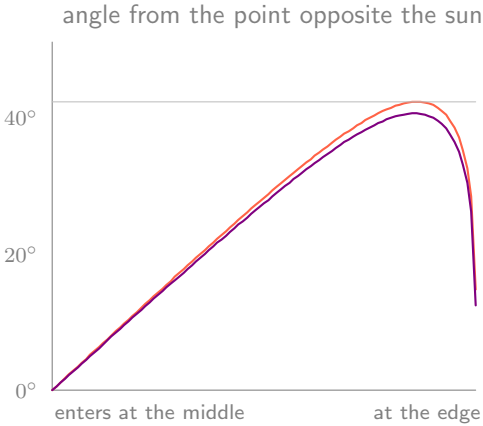


- A highest angle: the light piles up there.



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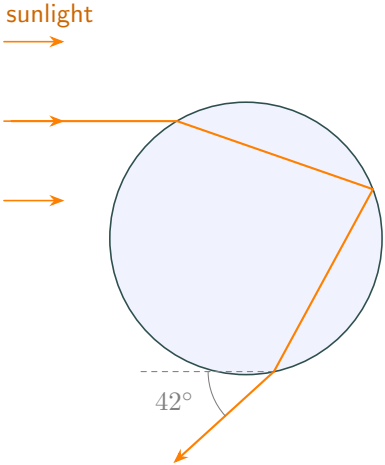

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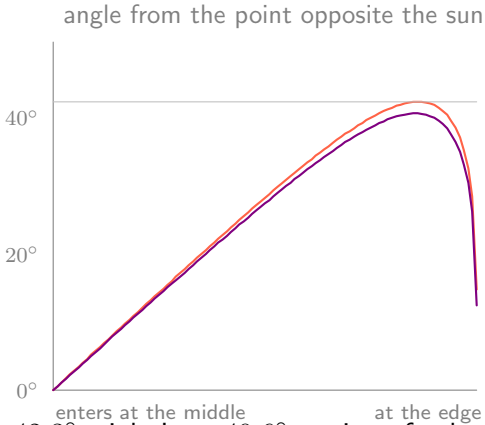

- Red's pile at $42.3°$, violet's at $40.6°$: a ring of colours.

Worked: the rainbow's angle

Try first: red light, $n = 1.3312$. Where is the bow?

angle out: $4\theta_t - 2\theta_i$; the top of the curve: $\cos^2 \theta_i = (n^2 - 1)/3$

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$\cos^2\theta_i = (1.7721 - 1)/3 = 0.2574$: $\theta_i = 59.51°$

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Snell: $\sin\theta_t = \sin 59.51°/1.3312 = 0.6474$: $\theta_t = 40.34°$

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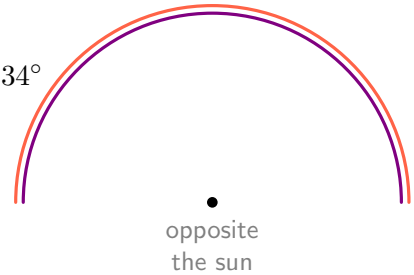
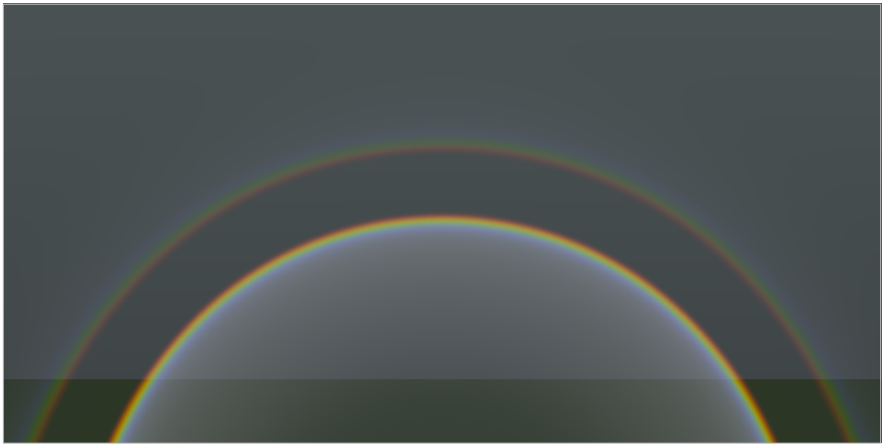
$4 \times 40.34° - 2 \times 59.51° = 42.34°$

violet, $n = 1.3430$: $40.64°$



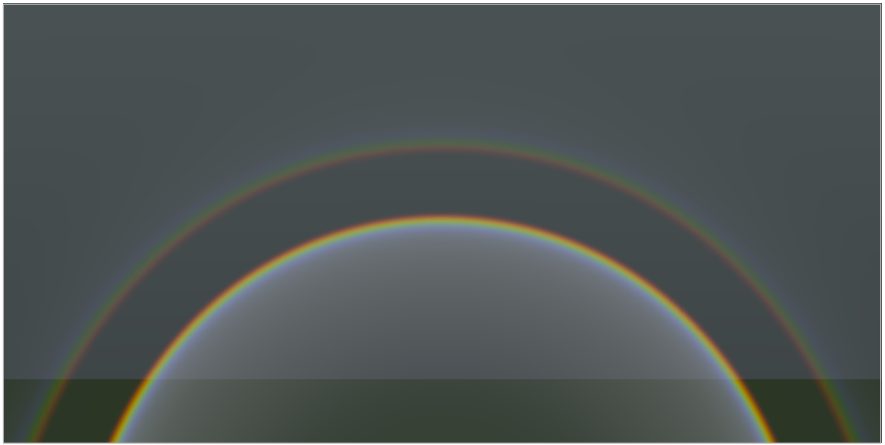
- Red outside, violet inside; the whole bow is $1.7°$ wide.

Rendered: a rainbow



rain in front of a dark cloud, the sun behind us, 22° up

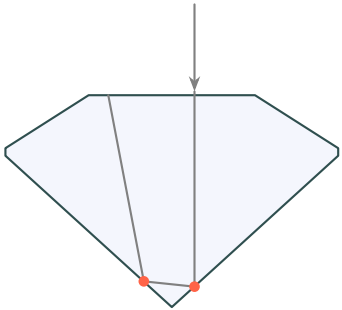
Rendered: a rainbow



rain in front of a dark cloud, the sun behind us, 22° up

- Two bows, the second's colours reversed; brighter inside, darker between.

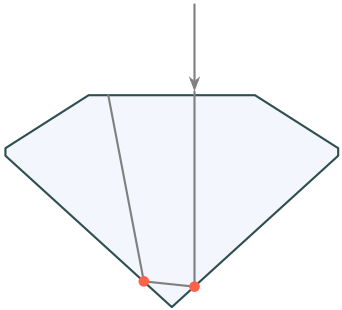
Diamond: trapped light, spread colours



a brilliant cut, seen from the side

	BK7 glass	diamond
all mirrored past	$41.2°$	$24.4°$

Diamond: trapped light, spread colours

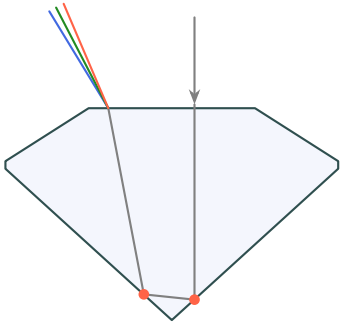


a brilliant cut, seen from the side

	BK7 glass	diamond
all mirrored past	$41.2°$	$24.4°$

- Lecture 18's total reflection starts at $24.4°$: light that enters bounces round and leaves through the top.

Diamond: trapped light, spread colours



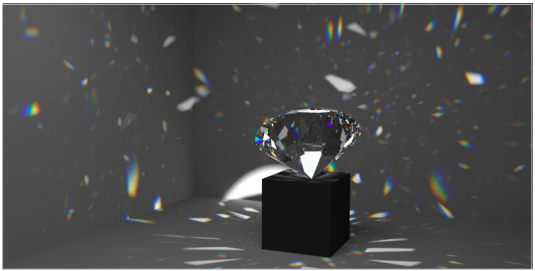
a brilliant cut, seen from the side

	BK7 glass	diamond
all mirrored past	$41.2°$	$24.4°$
n, 400 to 700 nm	-0.018	-0.058

- Lecture 18's total reflection starts at $24.4°$: light that enters bounces round and leaves through the top.
- Three times glass's spread: the colours leave apart. That is the fire.

Rendered: a diamond's fire

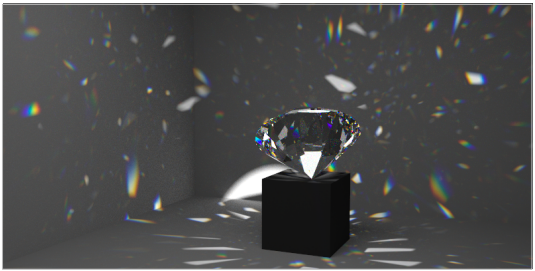
a narrow beam that falls only on the stone; the rest of the room is dim



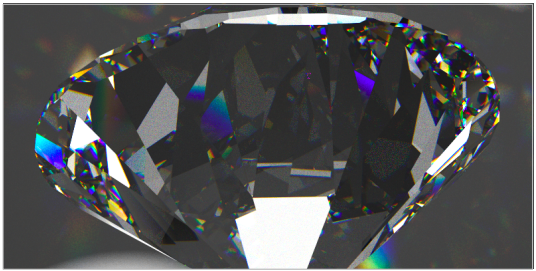
the corner

Rendered: a diamond's fire

a narrow beam that falls only on the stone; the rest of the room is dim



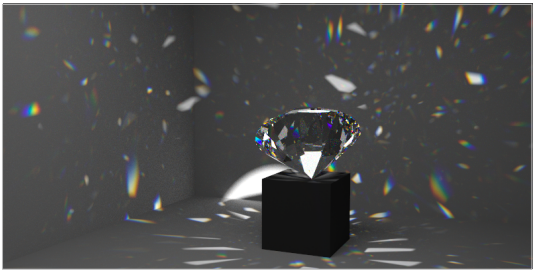
the corner



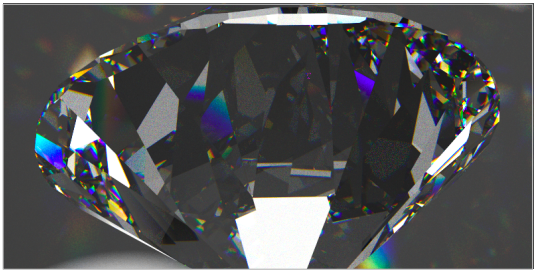
the stone, close up

Rendered: a diamond's fire

a narrow beam that falls only on the stone; the rest of the room is dim



the corner



the stone, close up

- Every coloured spot on the walls is the beam, split by the stone: its fire.
- Close up, the same split: coloured flashes at the crown's edge.

Quick check 2: bending by colour

Snell: $n_i \sin\theta_i = n_t \sin\theta_t$**; total reflection past** $\sin\theta = 1/n$**.**

1. BK7: $n(450) = 1.525$, $n(650) = 1.515$. Which is bent more?
2. Primary bow: red $42.3°$, violet $40.6°$. Which is outside?
3. Water, $n = 1.333$, light from air at $60°$: θ_t?
4. At $30°$ inside: glass or diamond, which mirrors it all?

- Each wavelength bends by its own index; the rest is lecture 18.

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2. Primary bow: red $42.3°$, violet $40.6°$. Which is outside?	red
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4. At $30°$ inside: glass or diamond, which mirrors it all?	diamond

- Each wavelength bends by its own index; the rest is lecture 18.

Recap: bending by colour

$n(\lambda) = A + B/\lambda^2$: blue bends more

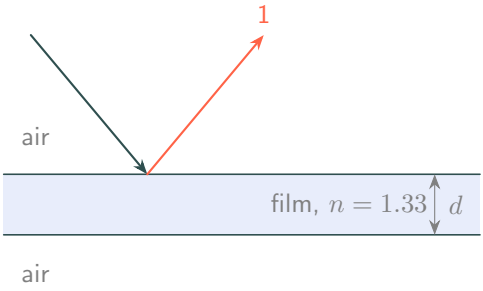
lecture 18's glass with `index(lam)`; heroes: follow one

rainbow: a pile-up at $42°$; diamond: $24.4°$ and three times the spread

Waves

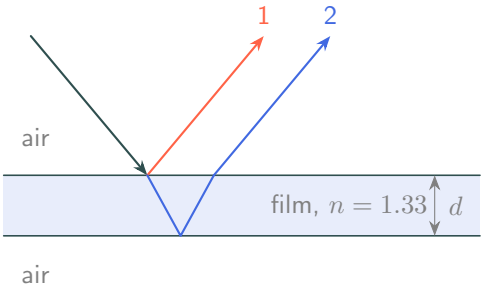
Colours that no ray can explain

Thin films: two reflections that meet



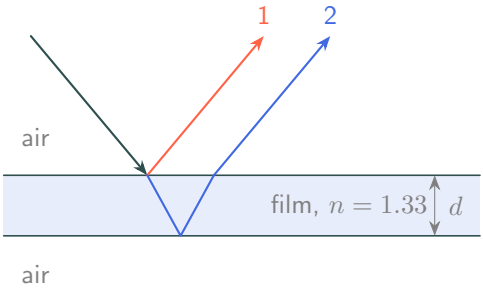
- Ray 1: mirrored off the top face.

Thin films: two reflections that meet



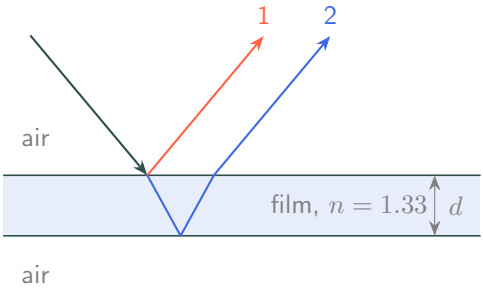
- Ray 1: mirrored off the top face.
- Ray 2: off the bottom face, then out.

Thin films: two reflections that meet



- Ray 1: mirrored off the top face.
- Ray 2: off the bottom face, then out.
- Ray 2 goes further: $2\,n\,d\cos\theta_t$

Thin films: two reflections that meet



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- Crest on crest: they add. Crest on trough: they cancel.

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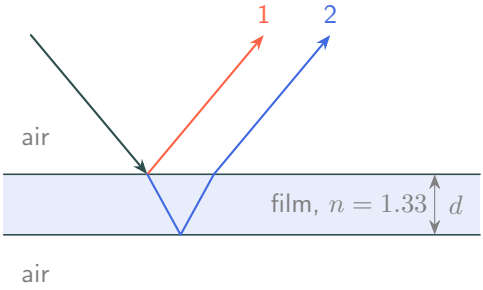

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  add: $2\,n\,d\cos\theta_t = (m + \frac{1}{2})\,\lambda$

  cancel: $2\,n\,d\cos\theta_t = m\,\lambda$

  soap in air; $m = 0, 1, 2, \ldots$

Worked: a soap film 300 nm thick

Try first: seen head-on, which colours cancel, which add?

$n = 1.33$, $d = 300$ nm, $\cos\theta_t = 1$: $2\,n\,d = 798$ nm

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cancel: $798 = m\lambda$: $\lambda = 798, 399$ nm

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cancel: $798 = m\lambda$: $\lambda = 798, 399$ nm

add: $798 = (m + \frac{1}{2})\lambda$: $\lambda = 532$ nm

(1596 and 319 nm: not visible)

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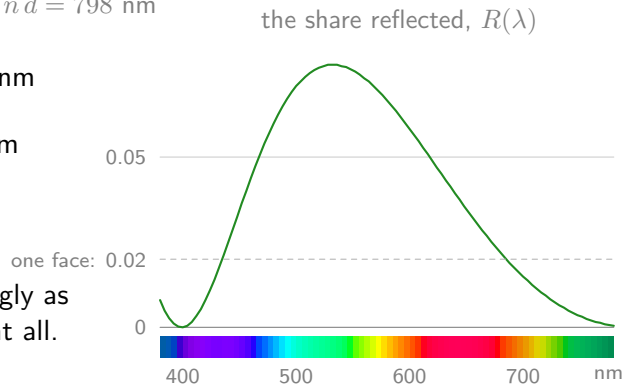
(1596 and 319 nm: not visible)

- Green reflected four times as strongly as one face; violet and deep red not at all.



Colour against thickness

the film's thickness d; seen head-on, in white light (brightened 12 times)



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- A film thinning as it drains runs through these colours; past 1000 nm they fade.

Colour against thickness

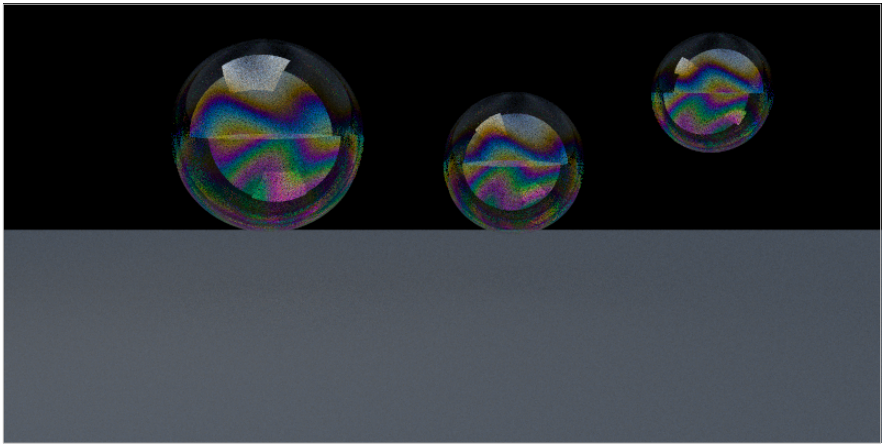
the film's thickness d; seen head-on, in white light (brightened 12 times)



- A film thinning as it drains runs through these colours; past 1000 nm they fade.
- In a tracer: the film's $R(\lambda, \theta, d)$ replaces lecture 18's Fresnel share.

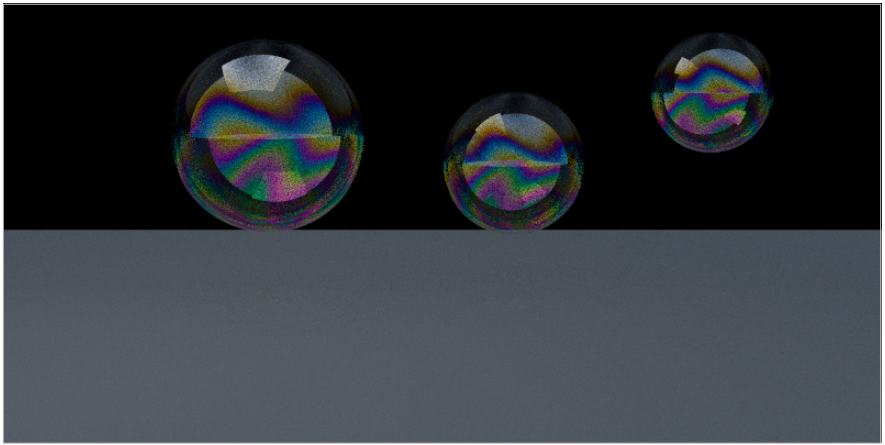
inside Extra 2's facets: Belcour and Barla 2017 (oil on metal, beetle shells, bubbles)

Rendered: soap bubbles



each wall of each bubble: reflect with chance $R(\lambda, \theta, d)$, else go straight on

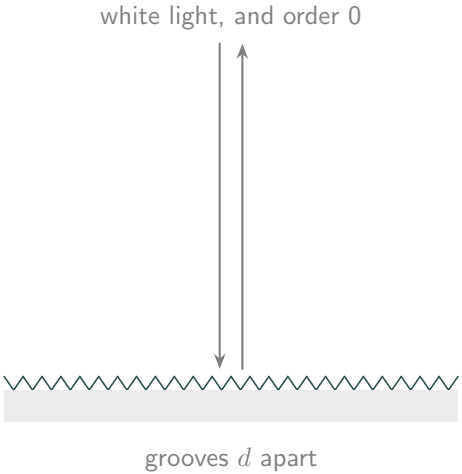
Rendered: soap bubbles



each wall of each bubble: reflect with chance $R(\lambda, \theta, d)$, else go straight on

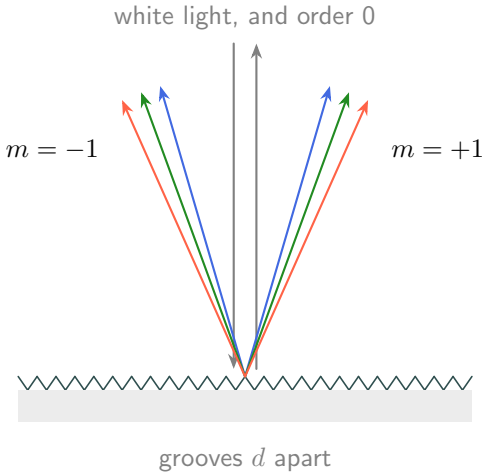
- Swirled bands of colour; the thin tops show the sky's reflection almost colourless.

Diffraction: grooves a wavelength apart



- Each groove sends out a little wave.

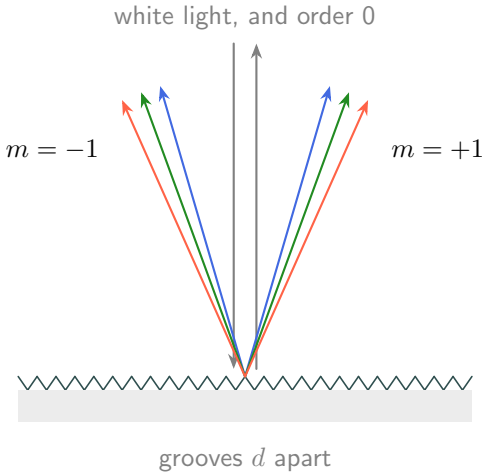
Diffraction: grooves a wavelength apart



- Each groove sends out a little wave.
- They add where each groove's wave is m whole wavelengths behind the next:

$$d \sin \theta_m = m \lambda$$

Diffraction: grooves a wavelength apart



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first order	450 nm	550 nm	650 nm
CD, 1.6 μm	$16.3°$	$20.1°$	$24.0°$
DVD, 0.74 μm	$37.5°$	$48.0°$	$61.4°$

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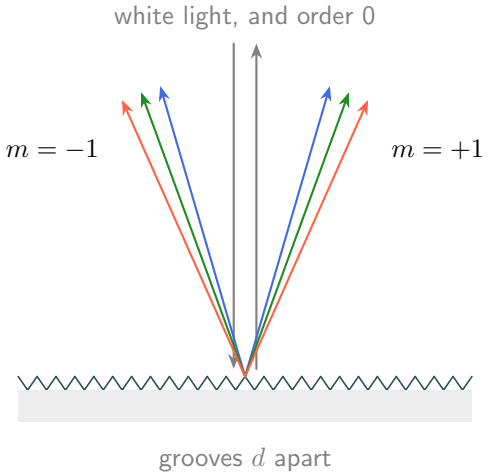

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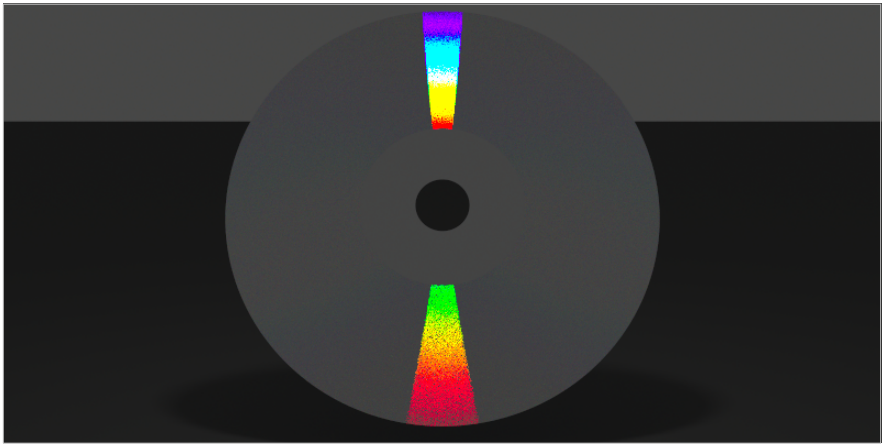
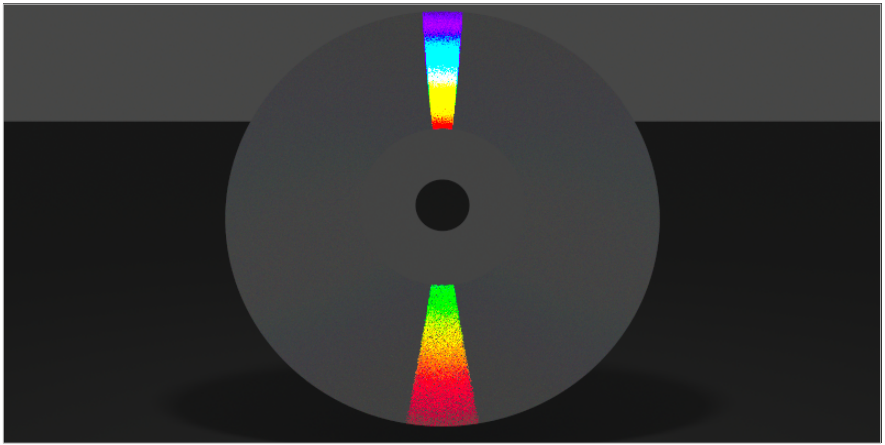
- A tracer: pick an order m, turn the mirror direction by $m\lambda/d$.

Rendered: a compact disc



one small lamp above the disc, out of the picture; the tracks are circles

Rendered: a compact disc

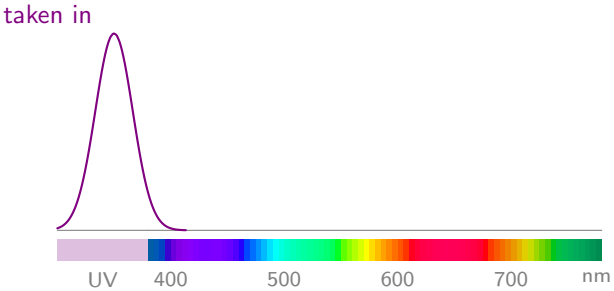


one small lamp above the disc, out of the picture; the tracks are circles

- Above the hole: the first order, violet outside; below: the second order.

Fluorescence: light that changes wavelength

- Some materials take in light of one wavelength . . .



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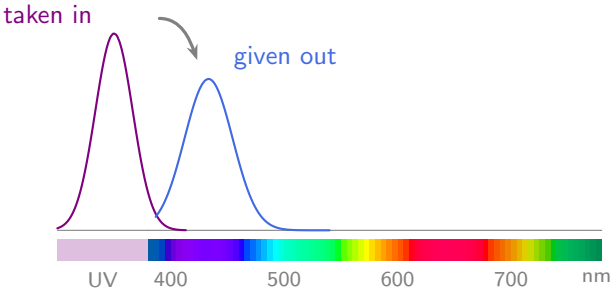
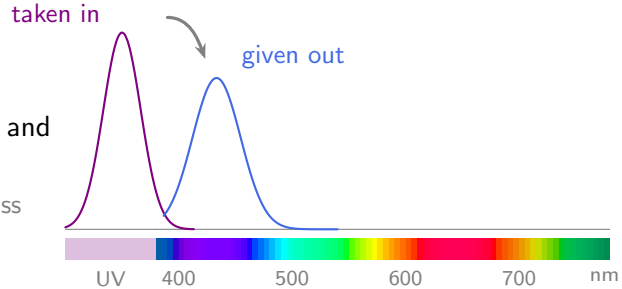


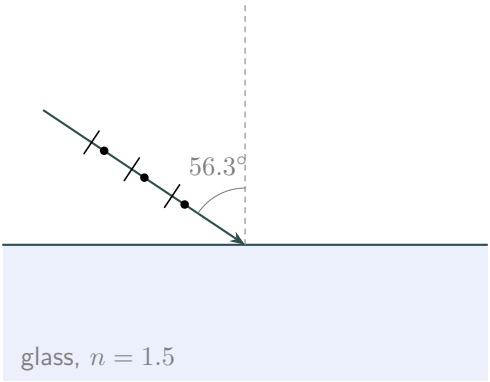
Fluorescence: light that changes wavelength

- Some materials take in light of one wavelength . . .
- . . . and give it out at a longer one: invisible ultraviolet in, blue out.
- In a tracer: the path's wavelength changes at the hit, by a table of in and out.

  heroes: the others drop out, as at glass

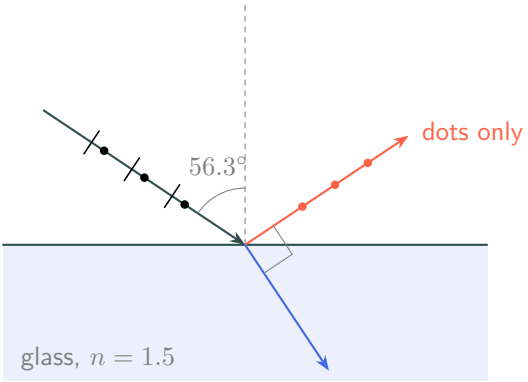


Polarisation: light that vibrates one way



- Light vibrates across its path; Fresnel's share differs for the two ways, s and p.

Polarisation: light that vibrates one way



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- At Brewster's angle, $\tan\theta = n$, $56.3°$ for glass, only s is mirrored.

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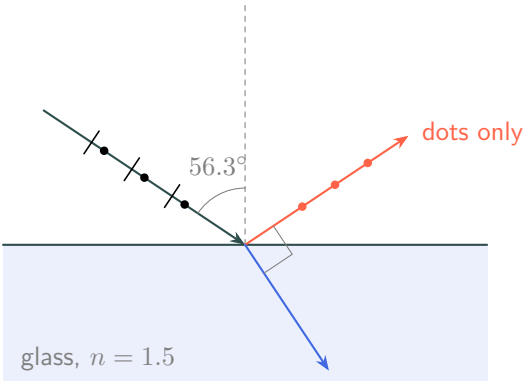

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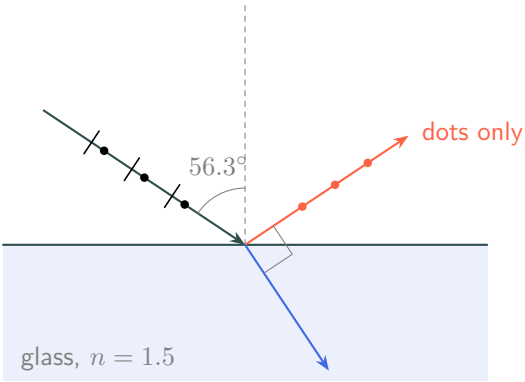


- Light vibrates across its path; Fresnel's share differs for the two ways, s and p.
- At Brewster's angle, $\tan\theta = n$, $56.3°$ for glass, only s is mirrored.
- Sunglasses that block s cut the glare off water and roads.
- A tracer that cares carries four numbers per wavelength (Stokes).

Recap: waves

film: two reflections; add at $2nd\cos\theta_t = (m + \frac{1}{2})\lambda$

grating: $d\sin\theta_m = m\lambda$; CD: 16° to 24°

fluorescence: λ changes; polarisation: s and p

Summary

1 Colour is a projection

$$X, Y, Z = \int L\,(\bar{x}, \bar{y}, \bar{z})\,d\lambda$$

metamers part under another lamp

2 Tracing wavelengths

one path adds $\dfrac{L(\lambda)\,(\bar{x}, \bar{y}, \bar{z})}{p(\lambda)}$

heroes: four λ's per path, weighed

3 Bending by colour

$n(\lambda)$: blue bends more

prism, rainbow at $42°$, diamond's fire

4 Waves

films and grooves: add or cancel

bubbles, CDs; fluorescence, polarisation

Reading

Pharr, Jakob and Humphreys, *Physically Based Rendering*, 4th edition (2023)

chapter 4: spectra, colour, sampled wavelengths, RGB to spectrum; free at pbr-book.org

Wilkie et al., "Hero wavelength spectral sampling", EGSR 2014

Jakob and Hanika, "A low-dimensional function space for efficient spectral upsampling", EGSR 2019

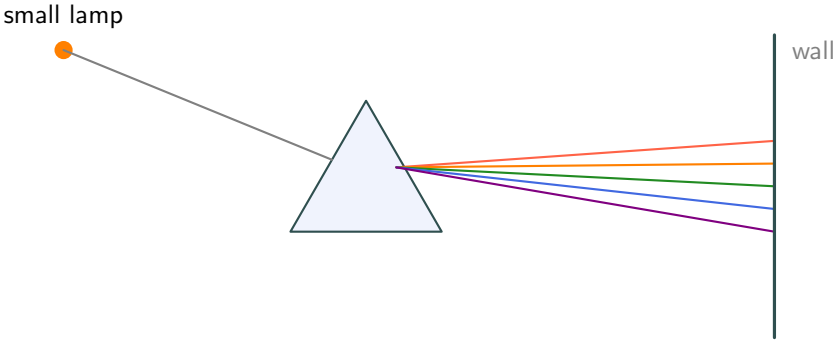
Belcour and Barla, "A practical extension to microfacet theory for the modeling of varying iridescence", SIGGRAPH 2017

Stam, "Diffraction shaders", SIGGRAPH 1999

Minnaert, *Light and Colour in the Outdoors*

rainbows, haloes and the colours of the sky, explained without a formula

Next: Extra 4, light paths



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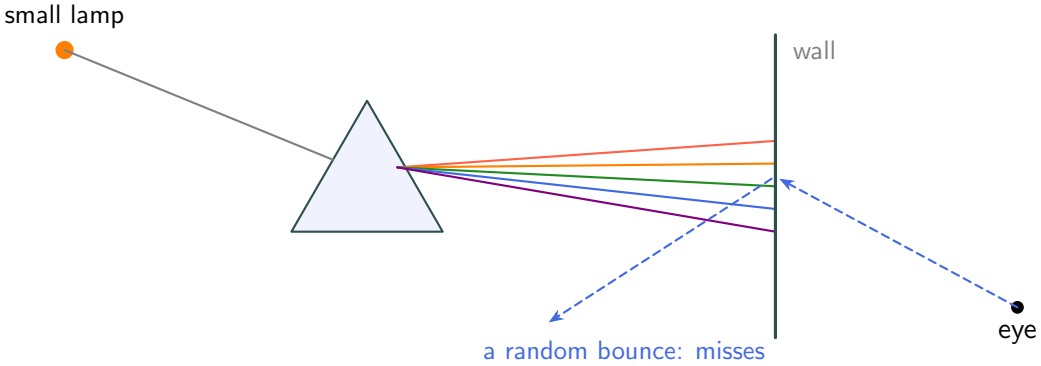

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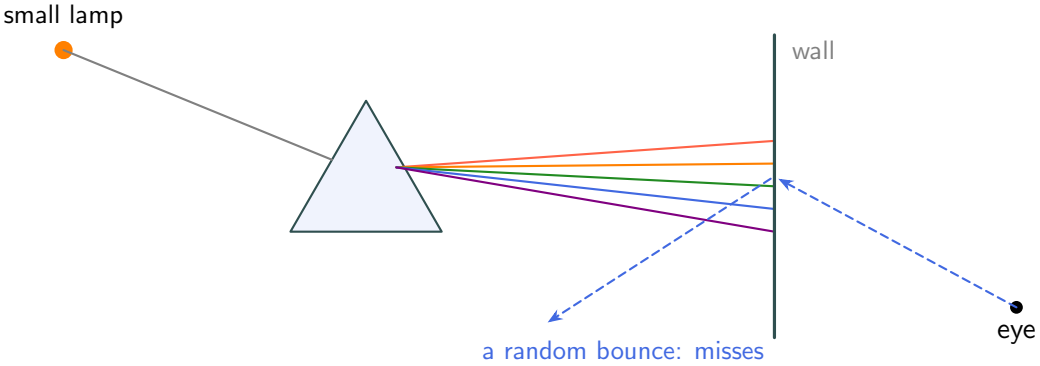

- Paths from the eye almost never find light through glass from a small lamp.

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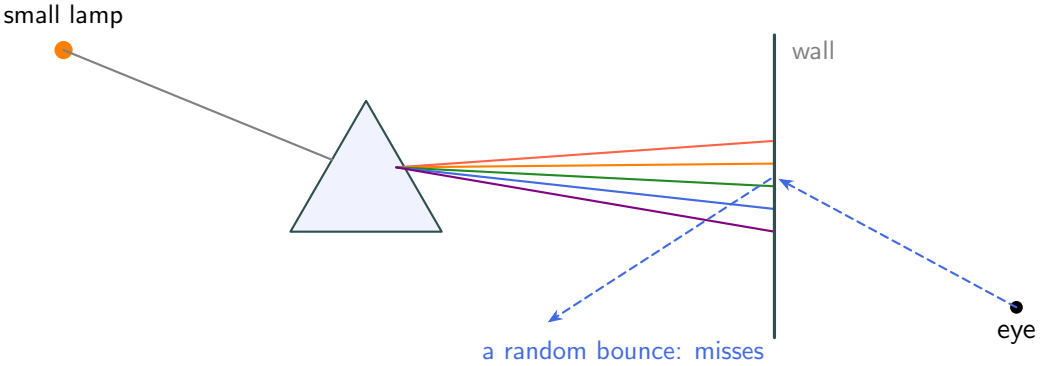

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- Extra 4: paths from the lamp too, photon maps, and Metropolis.