Extra 1: Better Sampling

Two Ways to Find the Lamp, and Both at Once

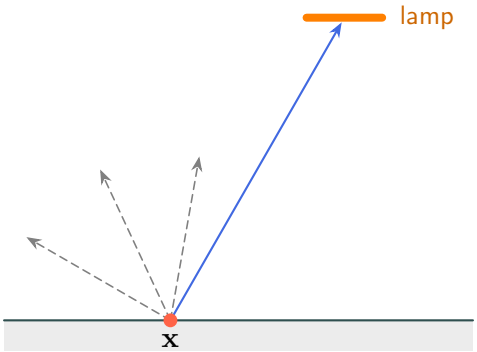
CS 453 — Computer Graphics

Fakhir Shaheen

Forman Christian University

Today: two ways to find the lamp, and both at once

Extra 1: self-study, not on the exams



way 1: a random bounce
happens to find it

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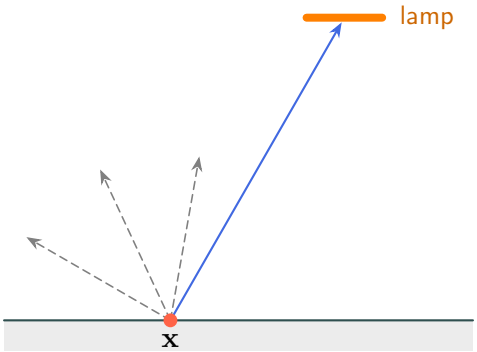

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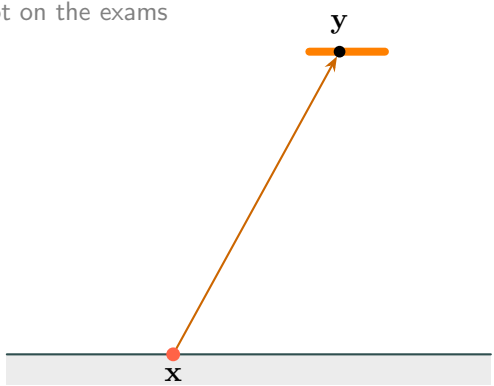

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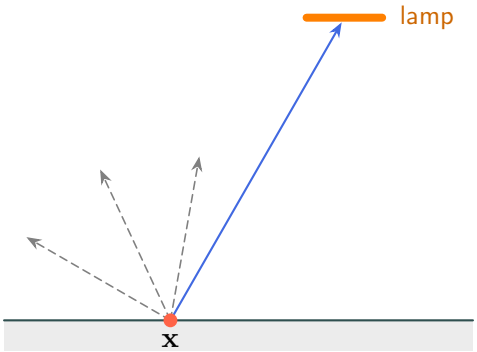

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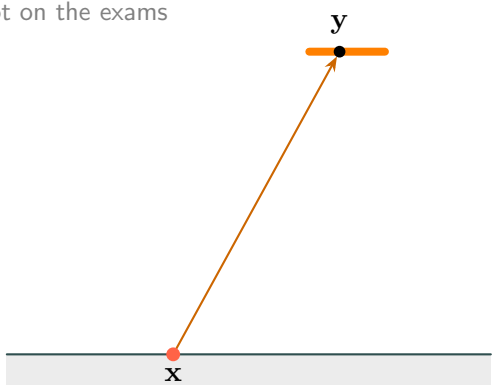

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- Each way fails in some scene.

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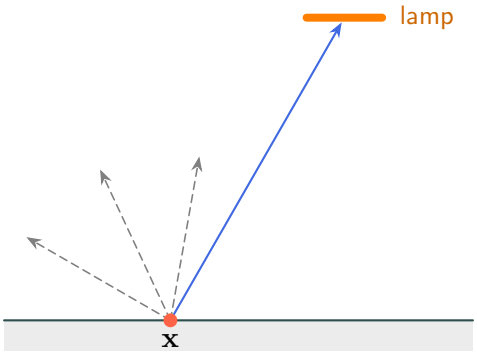
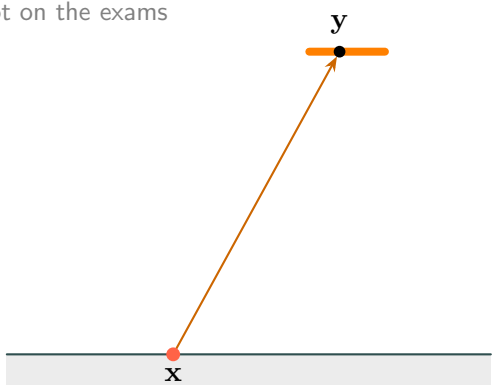


way 1: a random bounce happens to find it

- Each way fails in some scene.



way 2: a shadow ray to a point picked on it

- Today: both at once, weighted.

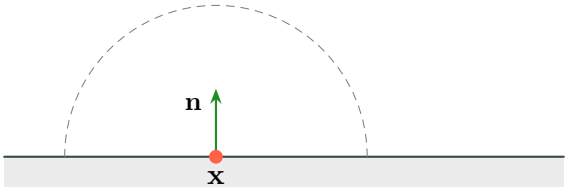
Two ways to find the lamp

The same integral, two kinds of pick

The lamp's light is an integral over its directions

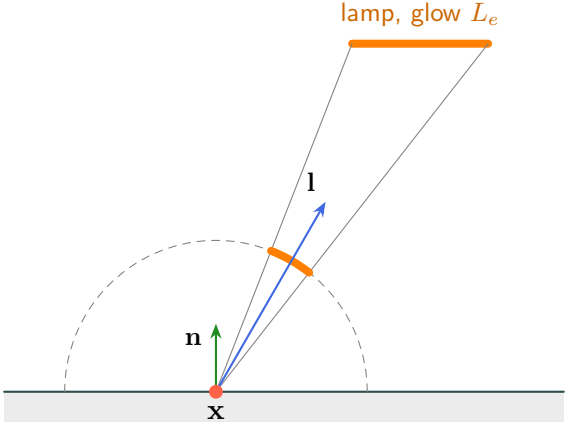
lamp, glow L_e

$$L_o = \int_{\text{lamp's directions}} \rho\, L_e \cos\theta \; d\sigma$$



wall light: the rest of the path (lecture 20)

The lamp's light is an integral over its directions



$$L_o = \int_{\text{lamp's directions}} \rho\, L_e \cos\theta \; d\sigma$$

- Only directions toward the lamp bring its light.

wall light: the rest of the path (lecture 20)

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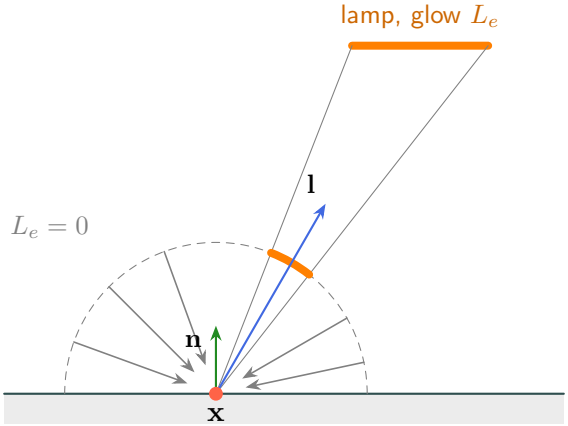
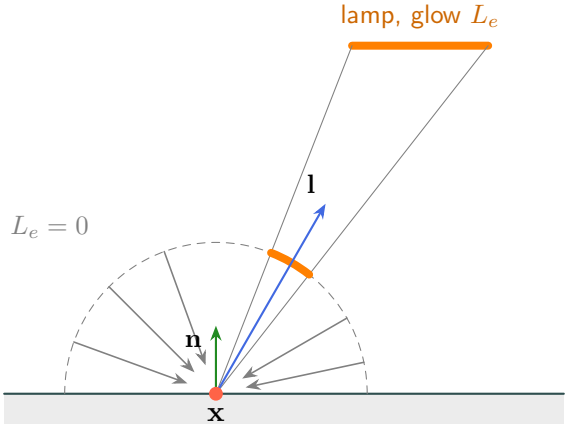


$$L_o = \int_{\text{lamp's directions}} \rho\, L_e \cos\theta \; d\sigma$$

- Only directions toward the lamp bring its light.
- The rest of the dome adds no lamp light.

wall light: the rest of the path (lecture 20)

The lamp's light is an integral over its directions

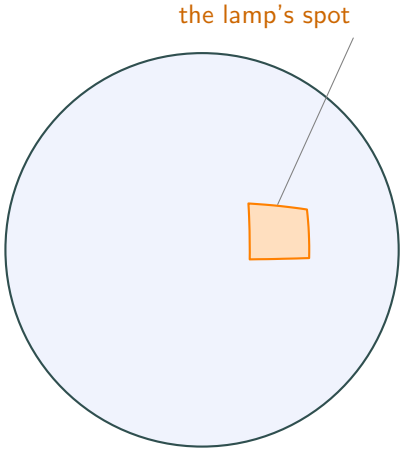


$$L_o = \int_{\text{lamp's directions}} \rho\, L_e \cos\theta \; d\sigma$$

- Only directions toward the lamp bring its light.
- The rest of the dome adds no lamp light.
- Monte Carlo: pick directions $\mathbf{l}$, average f/p.

wall light: the rest of the path (lecture 20)

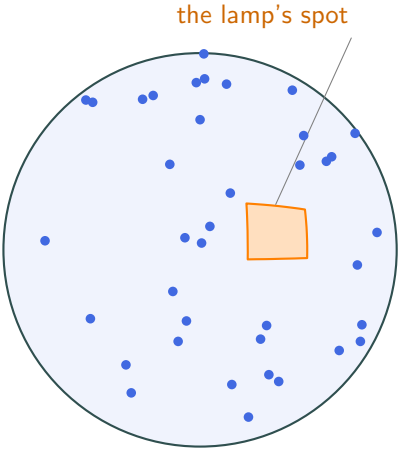
Way 1: pick by the BRDF, and hope to hit the lamp



the dome seen from above: area π

- Lecture 20's cosine picks fall evenly on the disc.

Way 1: pick by the BRDF, and hope to hit the lamp



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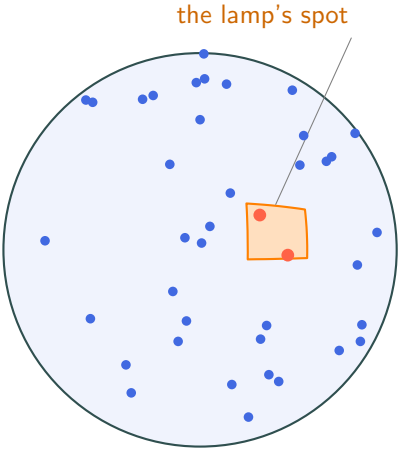
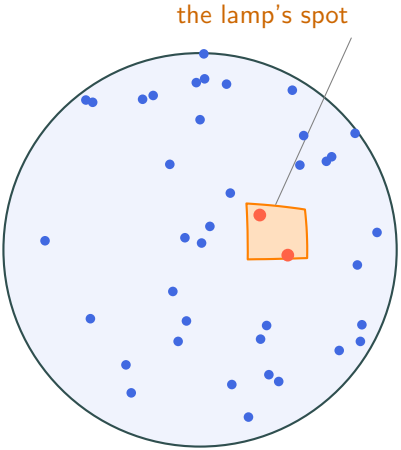


the dome seen from above: area π
here 2 of 40 picks hit; the spot is 2.6 % of the disc

- Lecture 20's cosine picks fall evenly on the disc.
- Chance of a hit: the spot's share of the disc.

Way 1: pick by the BRDF, and hope to hit the lamp

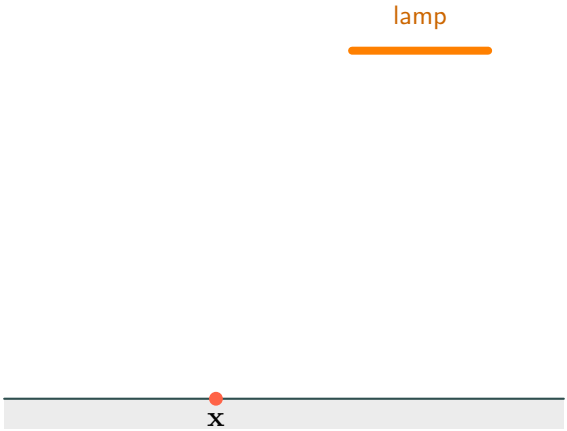


the dome seen from above: area π
here 2 of 40 picks hit; the spot is 2.6 % of the disc

- Lecture 20's cosine picks fall evenly on the disc.
- Chance of a hit: the spot's share of the disc.
- A hit brings back $R\,L_e$. A miss brings back no lamp light.

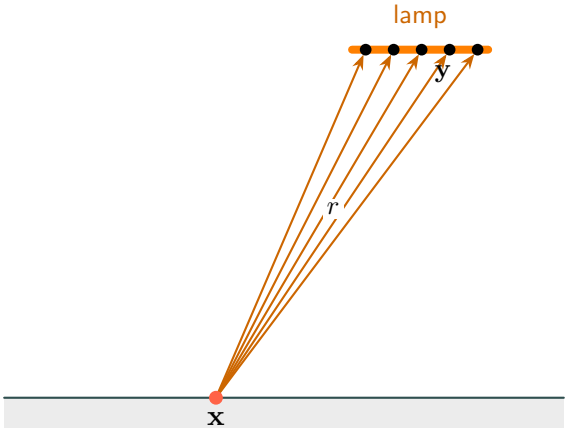
$$\frac{\rho\,L_e\cos\theta}{\cos\theta/\pi} = R\,L_e \qquad \text{(lecture 20)}$$

Way 2: pick a point on the lamp



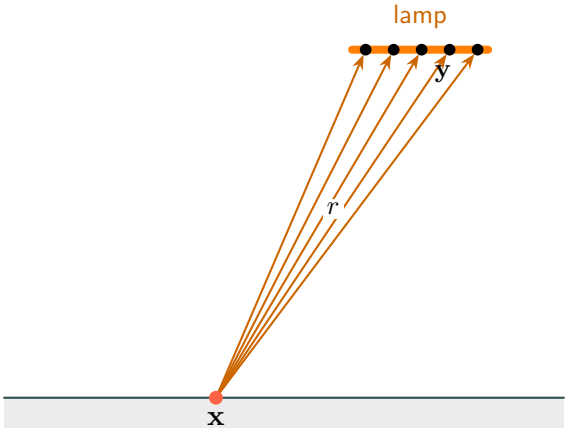
- Pick an even point $\mathbf{y}$ on the lamp.

Way 2: pick a point on the lamp



- Pick an even point $\mathbf{y}$ on the lamp.
- Every shadow ray reaches the lamp, unless blocked.

Way 2: pick a point on the lamp

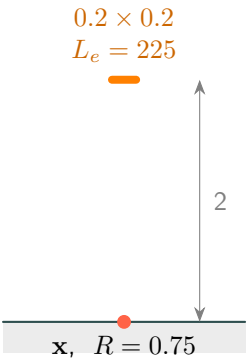


- Pick an even point $\mathbf{y}$ on the lamp.
- Every shadow ray reaches the lamp, unless blocked.
- A small lamp acts like lecture 16's point lamp, $I = L_e A$.

$$L_o \approx \frac{R}{\pi} \frac{L_e A \cos\theta}{r^2}$$

A: the lamp's area (exact: part two)

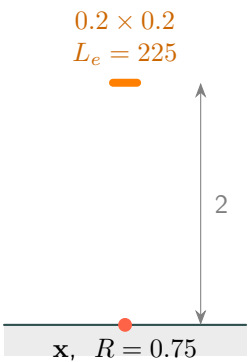
Worked: a small lamp, both ways



way	one pick brings back	how often
1: by the BRDF		
2: on the lamp		

- Try first: fill in the table.

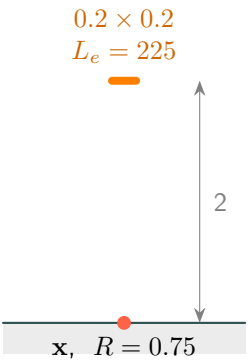
Worked: a small lamp, both ways



way	one pick brings back	how often
1: by the BRDF	$R\,L_e = 168.75$,	or 0
2: on the lamp		

- Try first: fill in the table.

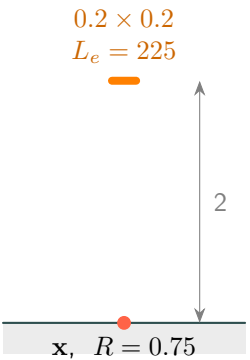
Worked: a small lamp, both ways



way	one pick brings back	how often
1: by the BRDF	$R\,L_e = 168.75$, or 0	1 pick in 315 $(A/r^2)/\pi = 0.01/\pi$
2: on the lamp		

- Try first: fill in the table.

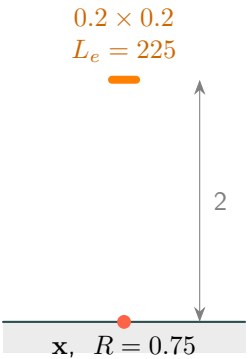
Worked: a small lamp, both ways



way	one pick brings back	how often
1: by the BRDF	$R\,L_e = 168.75$, or 0	1 pick in 315 $(A/r^2)/\pi = 0.01/\pi$
2: on the lamp	$\frac{0.75}{\pi} \cdot \frac{225 \times 0.04}{4} = 0.537$	every pick

- Try first: fill in the table.

Worked: a small lamp, both ways



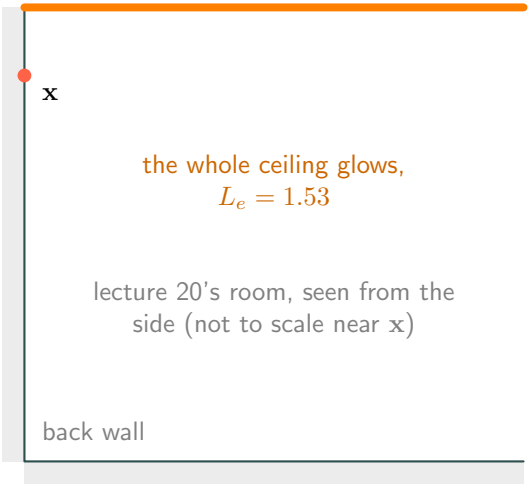
way	one pick brings back	how often
1: by the BRDF	$R\,L_e = 168.75$, or 0	1 pick in 315
		$(A/r^2)/\pi = 0.01/\pi$
2: on the lamp	$\frac{0.75}{\pi} \cdot \frac{225 \times 0.04}{4} = 0.537$	every pick

a corner point: 0.532

- Try first: fill in the table.
- Way 1: 314 zeros and one 168.75. Mean 0.535.
- Way 2: every pick within 1 % of 0.535.

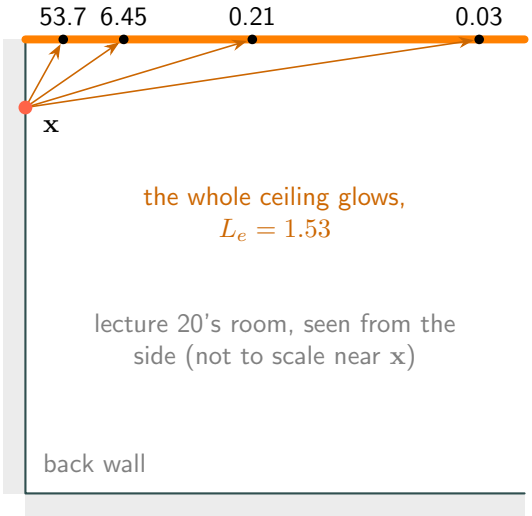
one pick each, 16 paths per pixel: way 1 is black with specks

Worked: the whole ceiling glows



Try first: which way is steadier here?

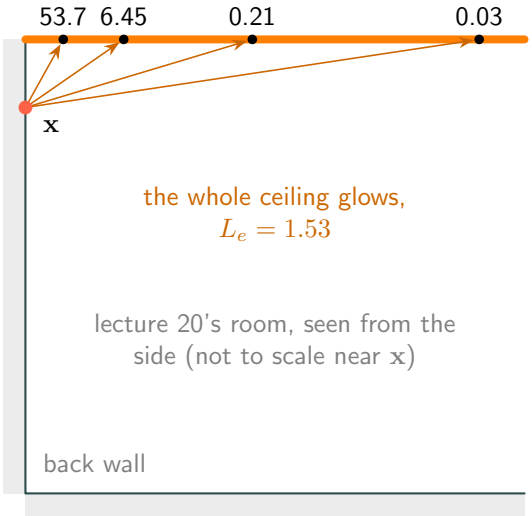
Worked: the whole ceiling glows



Try first: which way is steadier here?

- Way 2: a pick close to $\mathbf{x}$ gives 53.7, a far one 0.03.

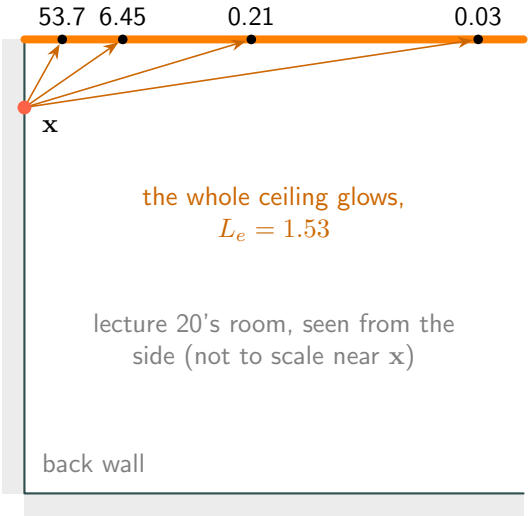
Worked: the whole ceiling glows



Try first: which way is steadier here?

- Way 2: a pick close to $\mathbf{x}$ gives 53.7, a far one 0.03.
- Way 1: about half the picks hit, each $R\,L_e = 1.15$.

Worked: the whole ceiling glows



Try first: which way is steadier here?

- Way 2: a pick close to $\mathbf{x}$ gives 53.7, a far one 0.03.
- Way 1: about half the picks hit, each $R\,L_e = 1.15$.
- Both average 0.54. Way 2's spread is five times way 1's.

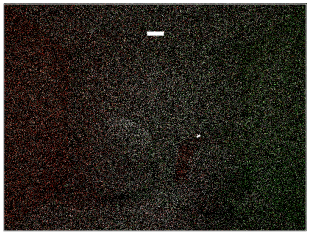
spread: way 1 0.57, way 2 3.1

Rendered: each way fails in one room

way 1: by the BRDF

way 2: on the lamp

small lamp



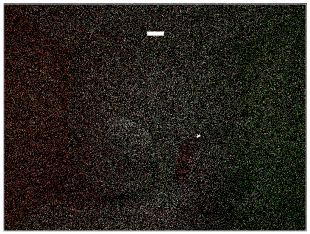
glowing ceiling

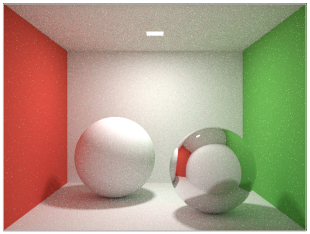
Rendered: each way fails in one room

way 1: by the BRDF

way 2: on the lamp

small lamp





glowing ceiling

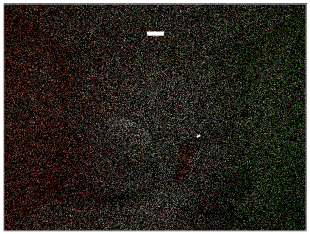
- Small lamp: way 2 wins.

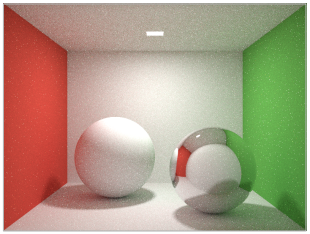
Rendered: each way fails in one room

way 1: by the BRDF

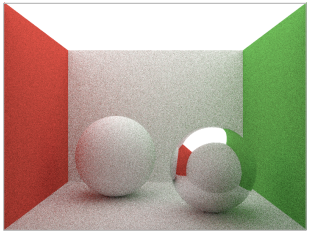
way 2: on the lamp

small lamp



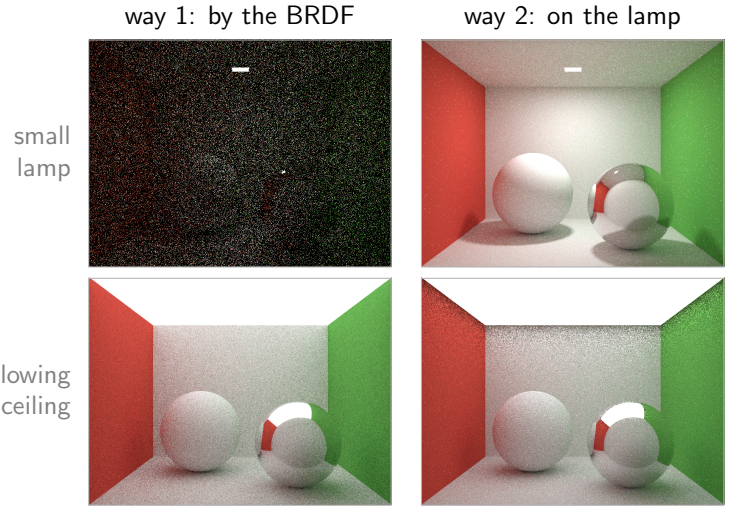


glowing ceiling



- Small lamp: way 2 wins.

Rendered: each way fails in one room



- Small lamp: way 2 wins.
- Big lamp: way 1 wins near it.

Quick check 1: which way, and how often?

A lamp's spot covers 2 % of lecture 19's disc. A hit brings back $R L_e = 50$.

1. Out of 100 cosine picks, how many hit the lamp, on average?
2. What does way 1 estimate, on average?
3. Lamp points 0.5 and 2 from $\mathbf{x}$, same angles: ratio of values?
4. A tiny bright bulb, far away: which way?

- Way 1 loses on small lamps, way 2 close to big ones.

Quick check 1: which way, and how often?

A lamp's spot covers 2 % of lecture 19's disc. A hit brings back $R\,L_e = 50$.

1.	Out of 100 cosine picks, how many hit the lamp, on average?	2
2.	What does way 1 estimate, on average?	$50 \times 0.02 = 1$
3.	Lamp points 0.5 and 2 from $\mathbf{x}$, same angles: ratio of values?	$4^2 = 16$
4.	A tiny bright bulb, far away: which way?	way 2

- Way 1 loses on small lamps, way 2 close to big ones.

Recap: two ways to find the lamp

way 1, by the BRDF: $R\,L_e$ or nothing; rare hits on a small lamp

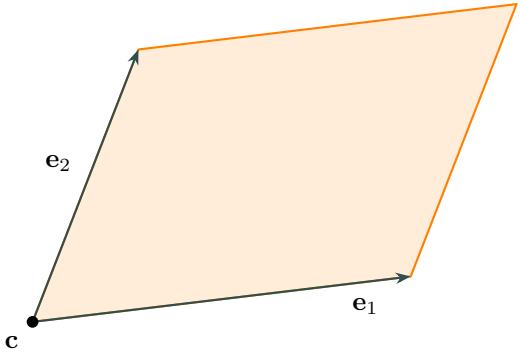
way 2, on the lamp: every pick counts; huge close to a big lamp

both are right on average

Area lamps, properly

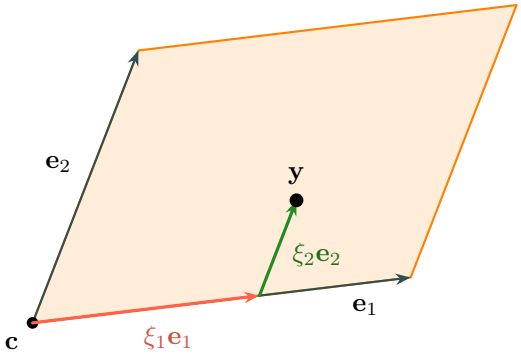
From a point on the lamp to a direction

An even point on a rectangle lamp



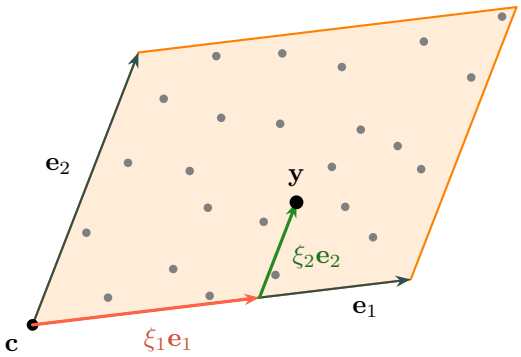
- A lamp: corner $\mathbf{c}$, edges $\mathbf{e}_1$ and $\mathbf{e}_2$.

An even point on a rectangle lamp



- A lamp: corner $\mathbf{c}$, edges $\mathbf{e}_1$ and $\mathbf{e}_2$.
- Two even random numbers ξ_1, ξ_2 in $[0, 1)$:
  $\mathbf{y} = \mathbf{c} + \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2$

An even point on a rectangle lamp

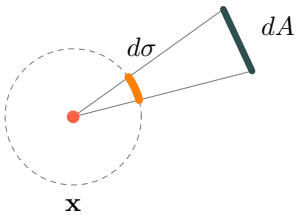


- A lamp: corner $\mathbf{c}$, edges $\mathbf{e}_1$ and $\mathbf{e}_2$.
- Two even random numbers ξ_1, ξ_2 in $[0, 1)$:
  $\mathbf{y} = \mathbf{c} + \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2$
- Every spot equally likely: per unit area,
  $p_A = \dfrac{1}{A}, \qquad A = |\mathbf{e}_1 \times \mathbf{e}_2|$

Seen from $\mathbf{x}$, a patch covers fewer directions

$$d\sigma = \frac{\cos\theta_y \, dA}{r^2}$$

a patch at distance r

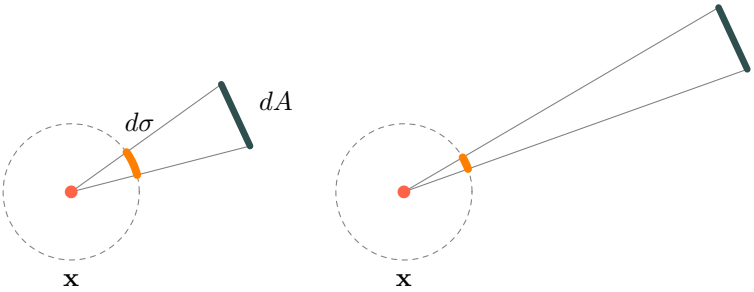


Seen from $\mathbf{x}$, a patch covers fewer directions

$$d\sigma = \frac{\cos\theta_y \, dA}{r^2}$$

a patch at distance r

twice as far: half as wide, in 3D a quarter: $1/r^2$



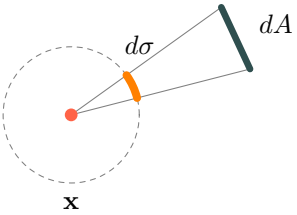
Seen from $\mathbf{x}$, a patch covers fewer directions

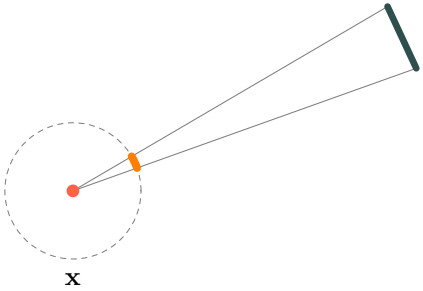
$$d\sigma = \frac{\cos\theta_y \, dA}{r^2}$$

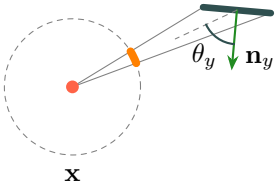
a patch at distance r

twice as far: half as wide, in 3D a quarter: $1/r^2$

turned by θ_y: $\cos\theta_y$ as much



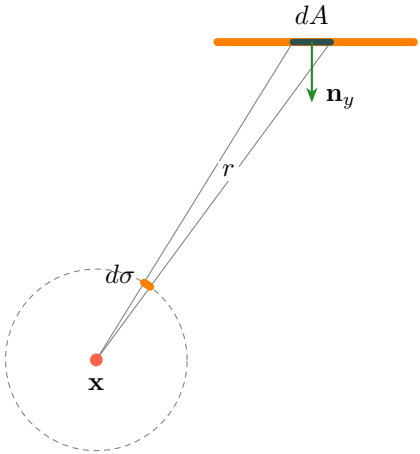




From area to directions: the density of a lamp pick

The chance of landing in the patch, counted two ways:

$$p_\sigma \, d\sigma \;=\; p_A \, dA$$



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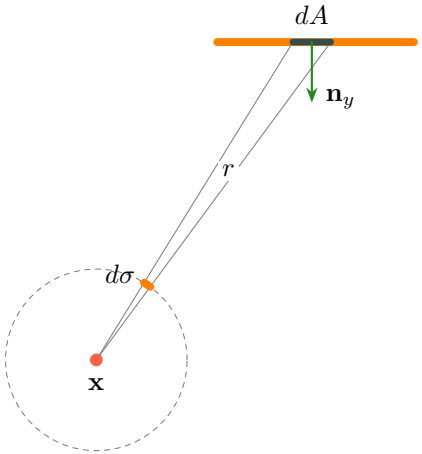

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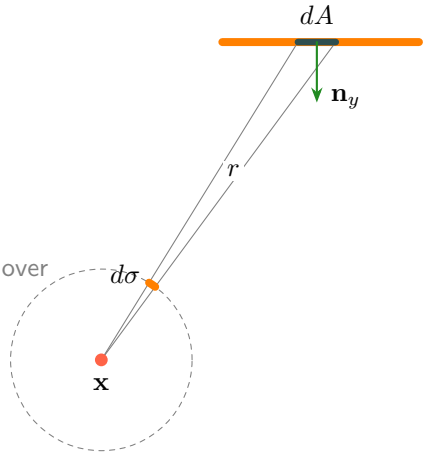
$$= \; \frac{1}{A} \cdot \frac{r^2}{\cos\theta_y} \qquad p_A = 1/A; \;\; \text{last slide, turned over}$$



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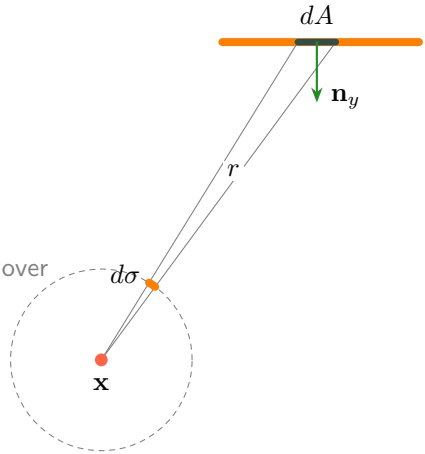
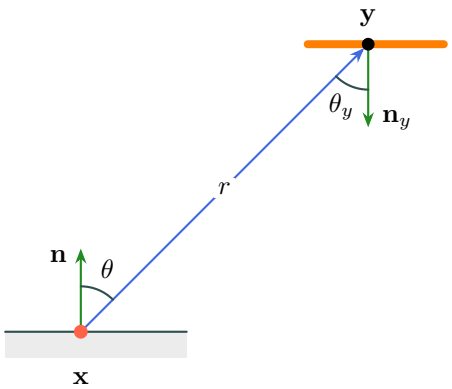
$$p_{\text{lamp}} = \frac{r^2}{A \cos\theta_y}$$

per unit of directions, for the direction to $\mathbf{y}$



What one point on the lamp is worth

$$\frac{f}{p} = \frac{\rho\, L_e \cos\theta}{p_{\text{lamp}}}$$



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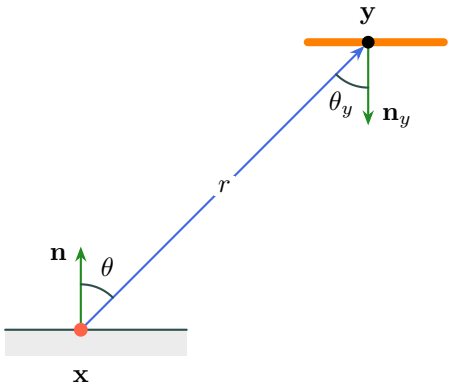

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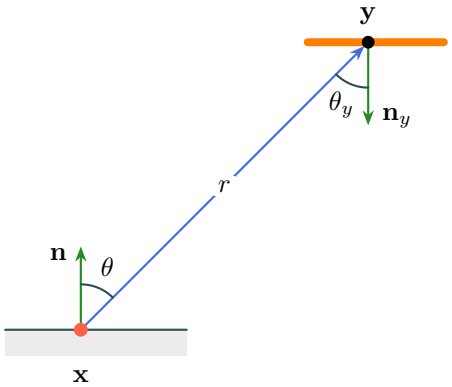

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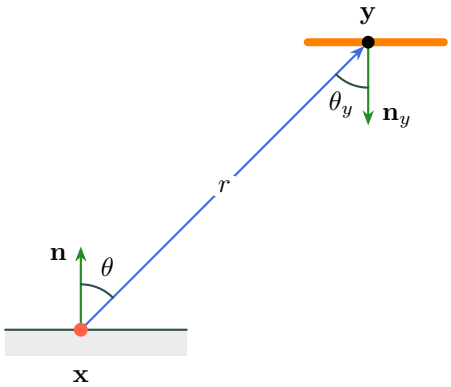
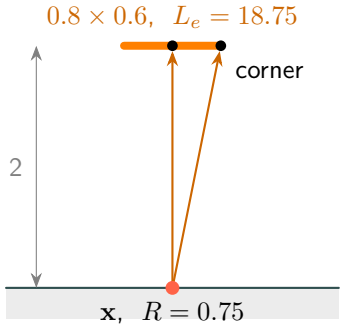
- A small lamp: $L_e A \cos\theta_y$ is lecture 16's I, and this is $\rho\, I \cos\theta / r^2$.
- $\cos\theta \le 0$ or $\cos\theta_y \le 0$: the lamp is behind $\mathbf{x}$, or $\mathbf{x}$ is behind the lamp. Add nothing.



Worked: lecture 20's panel, two picks

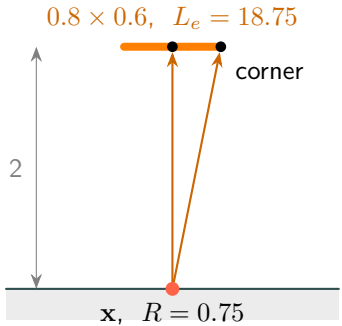


corner: 0.4 across, 0.3 deep

Try first: $\mathbf{y}$ at the centre, and at a corner.

pick	r^2	$\cos\theta, \ \cos\theta_y$	$\frac{L_e A \cos\cos}{r^2}$	$\times \frac{R}{\pi}$
centre				
corner				

Worked: lecture 20's panel, two picks



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Try first: $\mathbf{y}$ at the centre, and at a corner.

pick	r^2	$\cos\theta$, $\cos\theta_y$	$\frac{L_e A \cos\cos}{r^2}$	$\times \frac{R}{\pi}$
centre	4	1, 1	2.25	0.537
corner				

- $L_e A = 9$: lecture 19's lamp, $I_0 = 9$.

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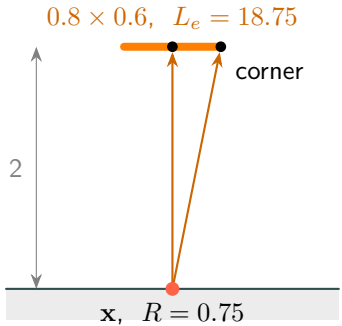

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centre	4	1, 1	2.25	0.537
corner	4.25	0.970, 0.970	1.99	0.476

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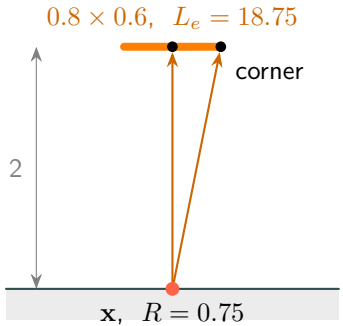

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pick	r^2	$\cos\theta$, $\cos\theta_y$	$\frac{L_e A \cos\cos}{r^2}$	$\times \frac{R}{\pi}$
centre	4	1, 1	2.25	0.537
corner	4.25	0.970, 0.970	1.99	0.476
exact			2.16	0.516

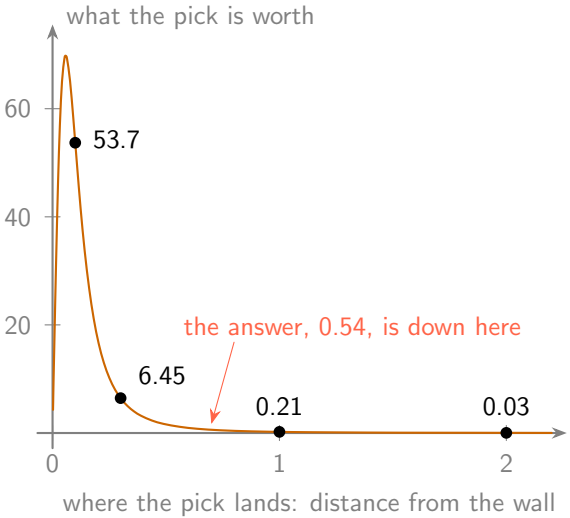
- $L_e A = 9$: lecture 19's lamp, $I_0 = 9$.
- Picks vary by 12 %: far from the lamp, way 2 is calm.

Close to a lamp, one over r^2 explodes



- The ceiling's wall point: one pick's worth, by where it lands.

Close to a lamp, one over r^2 explodes



- The ceiling's wall point: one pick's worth, by where it lands.
- Picks near $\mathbf{x}$ are rare and huge.

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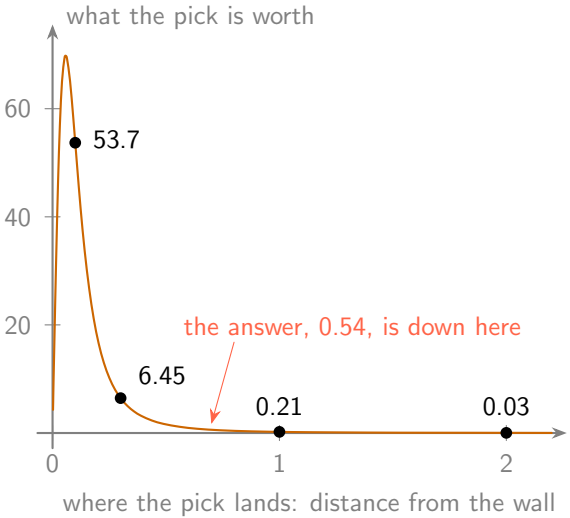

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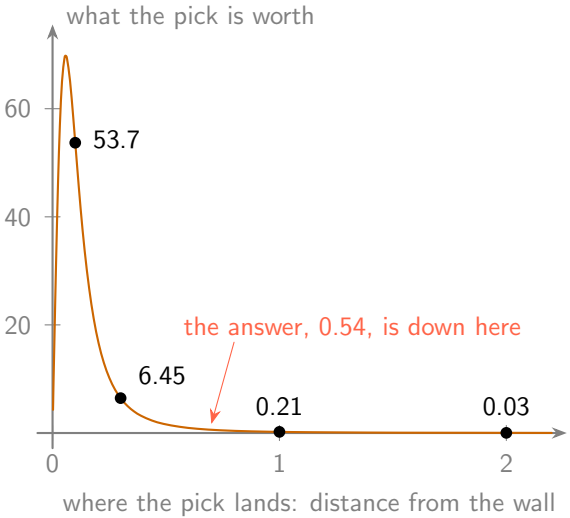
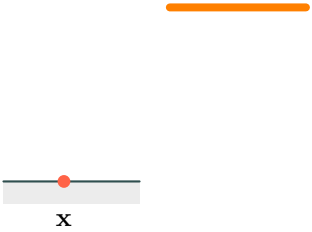


- The ceiling's wall point: one pick's worth, by where it lands.
- Picks near $\mathbf{x}$ are rare and huge.
- As $r \to 0$, $1/r^2$ grows without limit.
- Way 1 picks directions: no r^2 to blow up.

A rectangle lamp, in Python

```
class Quad:
    def __init__(self, corner, e1, e2, material):
        self.corner, self.e1, self.e2 = corner, e1, e2
        self.area = e1.cross(e2).length()
        self.n = e1.cross(e2) / self.area
```



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        xi1, xi2 = random.random(), random.random()
        return self.corner + xi1 * self.e1 + xi2 * self.e2
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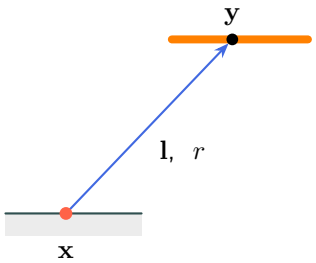
def lamp_share(lamp, x, n, material):
    y = lamp.sample()
    r = (y - x).length()
    l = (y - x) / r
```



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    l = (y - x) / r
    cos_x, cos_y = n.dot(l), -lamp.n.dot(l)
    if cos_x <= 0 or cos_y <= 0:  return Vec3(0,0,0)
```

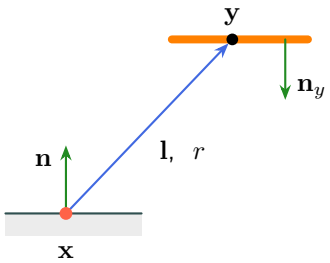

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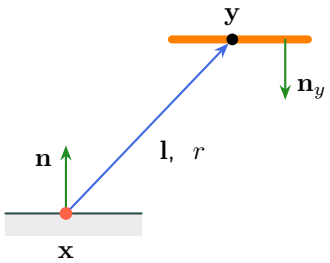


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    if scene.hit(Ray(x, l), 0.00001, r - 0.00001).t < math.inf:  return Vec3(0,0,0)
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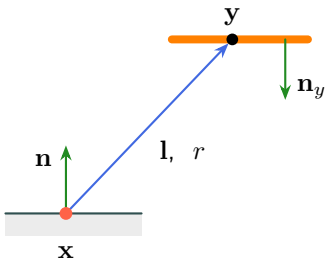


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    if scene.hit(Ray(x, l), 0.00001, r - 0.00001).t < math.inf:  return Vec3(0,0,0)
    k = cos_x * cos_y * lamp.area / r**2
    return material.albedo / math.pi * lamp.material.emission * k
```

Recap: area lamps, properly

$$\mathbf{y} = \mathbf{c} + \xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2, \qquad p_{\text{lamp}} = \frac{r^2}{A \cos\theta_y}$$

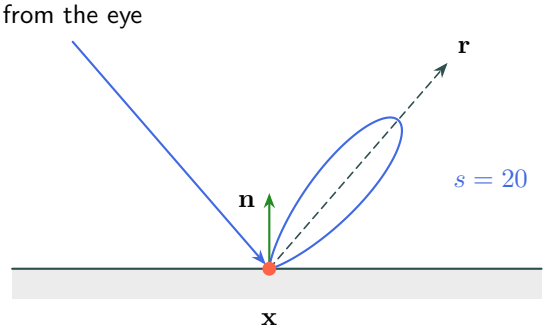
one pick: $\rho \, L_e \, \dfrac{A \cos\theta \, \cos\theta_y}{r^2}$

calm far from the lamp; wild close to it

Multiple importance sampling

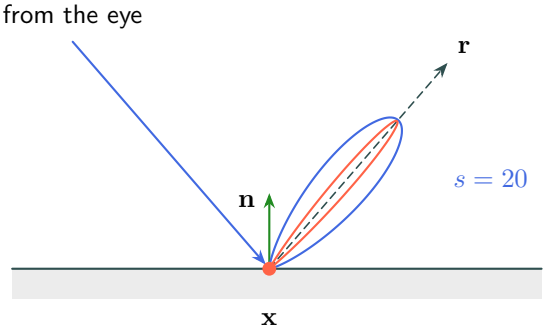
Both ways at once, weighted

A shiny surface: Phong's lobe as a BRDF



- A blurred mirror sends light near $\mathbf{r}$.

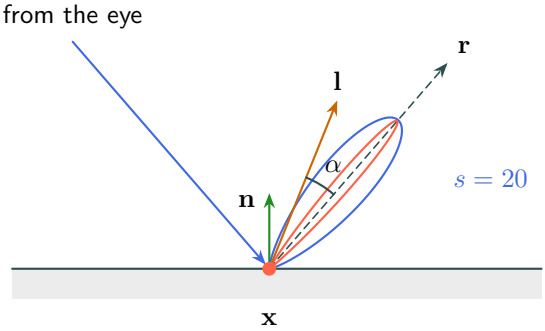
A shiny surface: Phong's lobe as a BRDF



- A blurred mirror sends light near $\mathbf{r}$.
- Large s (red, $s = 200$): a narrow lobe, a sharper mirror.

A shiny surface: Phong's lobe as a BRDF

$$\rho = R \frac{s+2}{2\pi} \cos^s \alpha$$



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- α: from $\mathbf{r}$ to the light's direction $\mathbf{l}$.

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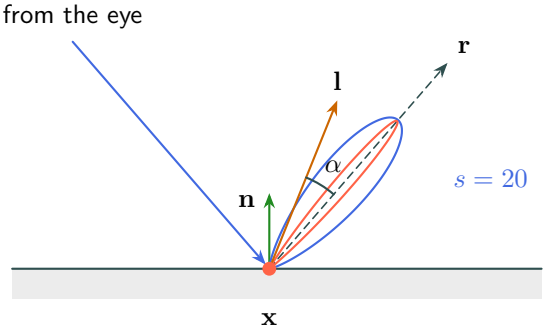

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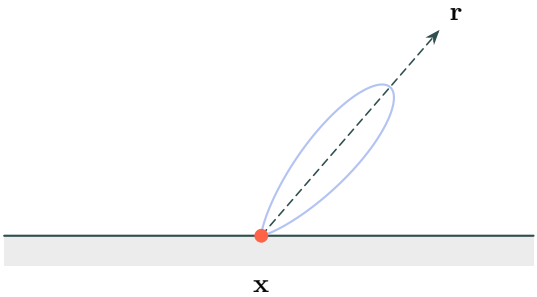
s: lecture 17's exponent; p is a density

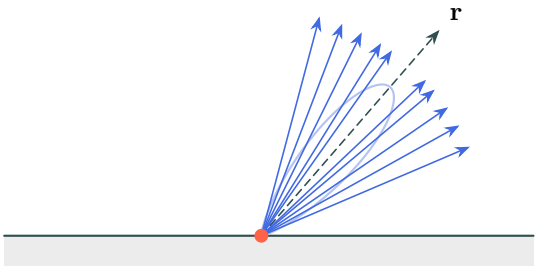
$(s+2)/2\pi$: the scale for a BRDF

Picking by the lobe



Pick α and a turn about $\mathbf{r}$:

Picking by the lobe

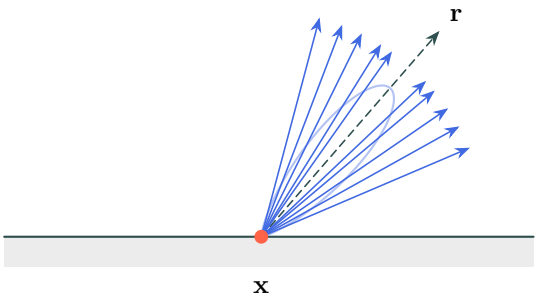


picks crowd where the lobe is large

Pick α and a turn about $\mathbf{r}$:

$$\cos\alpha = \xi_1^{1/(s+1)}, \quad \text{turn} = 2\pi\,\xi_2$$

Picking by the lobe



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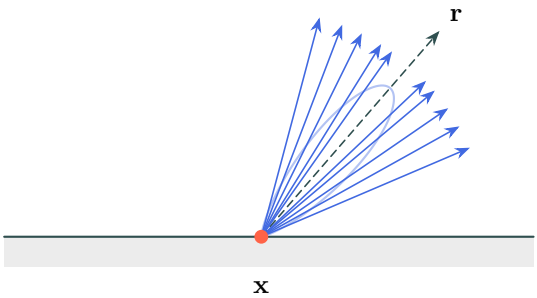

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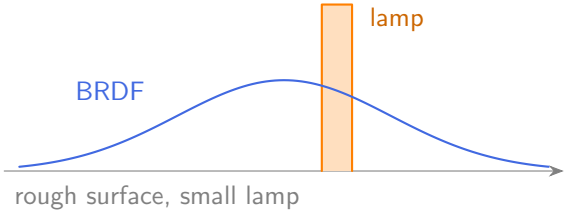
$$p_{\text{BRDF}} = \frac{s+1}{2\pi}\cos^s\alpha$$

$$\frac{\rho\, L_e \cos\theta}{p_{\text{BRDF}}} = R\,\frac{s+2}{s+1}\,L_e\cos\theta$$

- No spikes: the lobe cancels.

The narrower one wins



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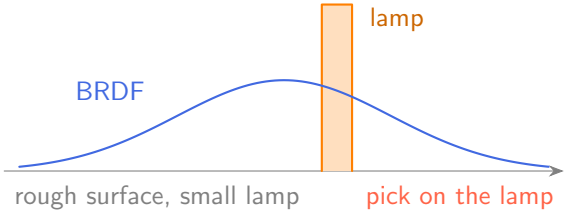

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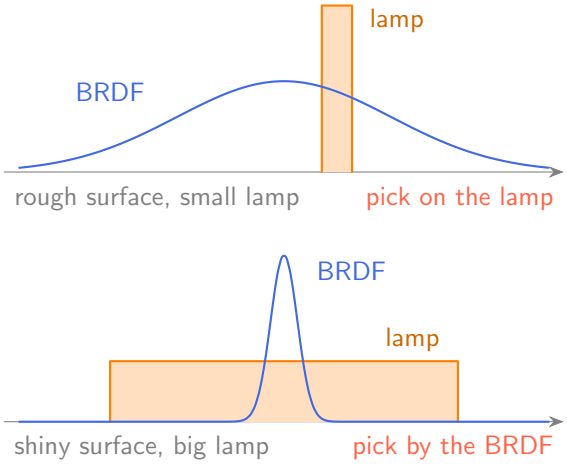
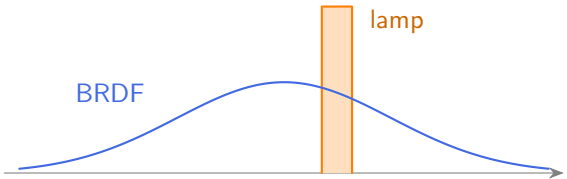
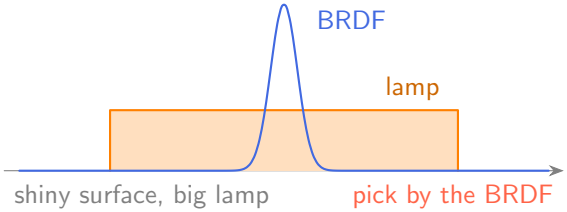


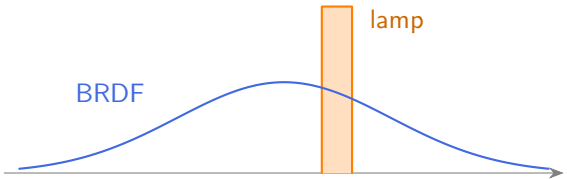
The narrower one wins

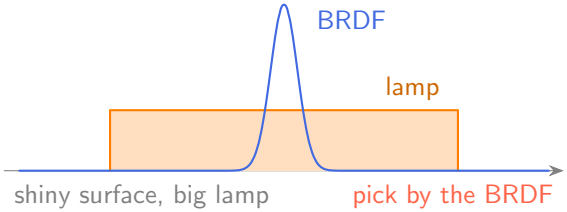




	rough	shiny
small lamp	lamp	about even
big lamp	about even	BRDF

The narrower one wins



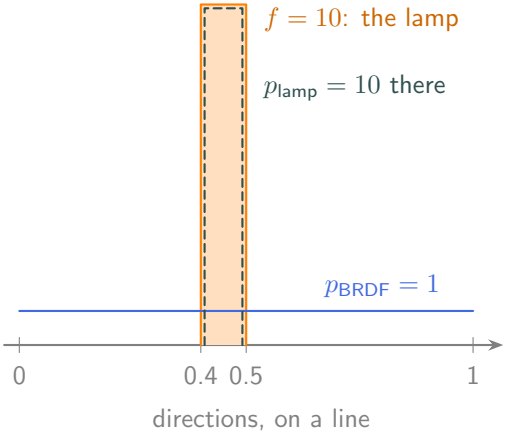


	rough	shiny
small lamp	lamp	about even
big lamp	about even	BRDF

- Pick by whichever covers fewer directions.
- One picture holds every case.

Worked: averaging the two ways keeps the specks

rough surface, small lamp: exact answer 1



Try first: what can each estimate be?

---

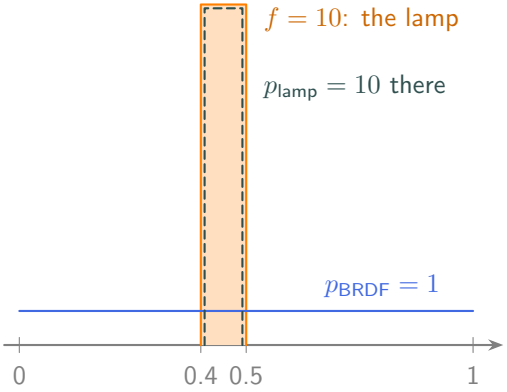
a BRDF pick

a lamp pick

both, averaged

Worked: averaging the two ways keeps the specks

rough surface, small lamp: exact answer 1

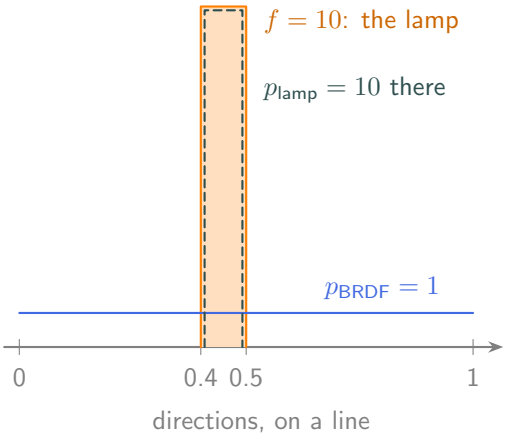


Try first: what can each estimate be?

a BRDF pick	10 (1 in 10), or 0
a lamp pick	
both, averaged	

Worked: averaging the two ways keeps the specks

rough surface, small lamp: exact answer 1

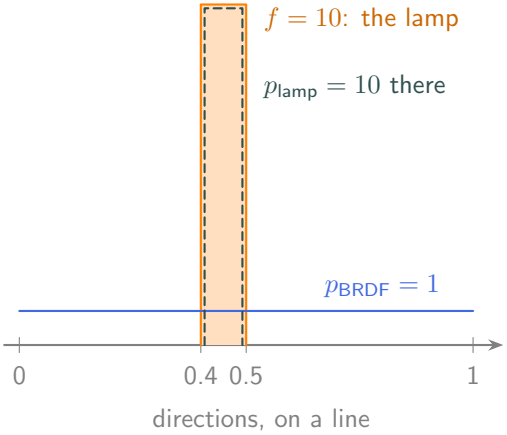


Try first: what can each estimate be?

a BRDF pick	10 (1 in 10), or 0
a lamp pick	1, always
both, averaged	

Worked: averaging the two ways keeps the specks

rough surface, small lamp: exact answer 1

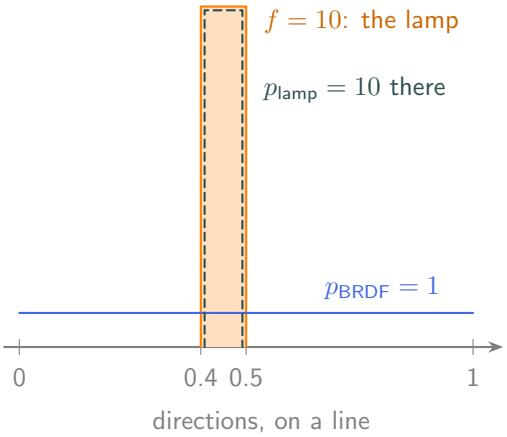


Try first: what can each estimate be?

a BRDF pick	10 (1 in 10), or 0
a lamp pick	1, always
both, averaged	5.5 (1 in 10), or 0.5

Worked: averaging the two ways keeps the specks

rough surface, small lamp: exact answer 1



Try first: what can each estimate be?

a BRDF pick	10 (1 in 10), or 0
a lamp pick	1, always
both, averaged	5.5 (1 in 10), or 0.5

- Half a speck is still a speck, and the calm way got worse.

Weights that add up to one keep it right

One pick each way, $\mathbf{l}_1$ by the BRDF and $\mathbf{l}_2$ on the lamp:

$$F = w_{\text{BRDF}}(\mathbf{l}_1)\,\frac{f(\mathbf{l}_1)}{p_{\text{BRDF}}(\mathbf{l}_1)} \;+\; w_{\text{lamp}}(\mathbf{l}_2)\,\frac{f(\mathbf{l}_2)}{p_{\text{lamp}}(\mathbf{l}_2)}$$

Weights that add up to one keep it right

One pick each way, $\mathbf{l}_1$ by the BRDF and $\mathbf{l}_2$ on the lamp:

$$F = w_{\text{BRDF}}(\mathbf{l}_1)\,\frac{f(\mathbf{l}_1)}{p_{\text{BRDF}}(\mathbf{l}_1)} \;+\; w_{\text{lamp}}(\mathbf{l}_2)\,\frac{f(\mathbf{l}_2)}{p_{\text{lamp}}(\mathbf{l}_2)}$$

$$\text{average of } F \;= \int w_{\text{BRDF}}\, f \;+\; \int w_{\text{lamp}}\, f$$

lecture 20: f/p averages to $\int f$

Weights that add up to one keep it right

One pick each way, $\mathbf{l}_1$ by the BRDF and $\mathbf{l}_2$ on the lamp:

$$F = w_{\text{BRDF}}(\mathbf{l}_1)\,\frac{f(\mathbf{l}_1)}{p_{\text{BRDF}}(\mathbf{l}_1)} \;+\; w_{\text{lamp}}(\mathbf{l}_2)\,\frac{f(\mathbf{l}_2)}{p_{\text{lamp}}(\mathbf{l}_2)}$$

$$\text{average of } F \;= \int w_{\text{BRDF}}\, f \;+\; \int w_{\text{lamp}}\, f$$

lecture 20: f/p averages to $\int f$

$$= \int f \qquad \text{when } w_{\text{BRDF}} + w_{\text{lamp}} = 1$$

Weights that add up to one keep it right

One pick each way, $\mathbf{l}_1$ by the BRDF and $\mathbf{l}_2$ on the lamp:

$$F = w_{\text{BRDF}}(\mathbf{l}_1)\,\frac{f(\mathbf{l}_1)}{p_{\text{BRDF}}(\mathbf{l}_1)} \;+\; w_{\text{lamp}}(\mathbf{l}_2)\,\frac{f(\mathbf{l}_2)}{p_{\text{lamp}}(\mathbf{l}_2)}$$

$$\text{average of } F \;= \int w_{\text{BRDF}}\, f \;+\; \int w_{\text{lamp}}\, f$$

lecture 20: f/p averages to $\int f$

$$= \int f \qquad \text{when } w_{\text{BRDF}} + w_{\text{lamp}} = 1$$

- Any weights that add up to one, direction by direction.

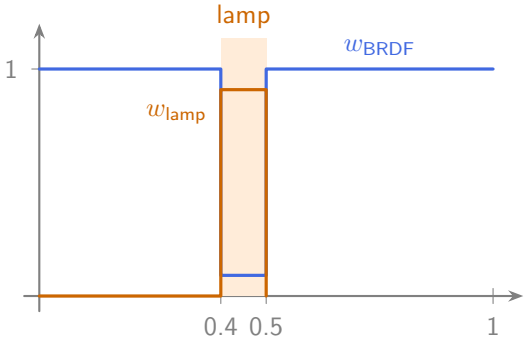
Veach's balance heuristic

$$w_{\text{BRDF}} = \frac{p_{\text{BRDF}}}{p_{\text{BRDF}} + p_{\text{lamp}}}, \qquad w_{\text{lamp}} = \frac{p_{\text{lamp}}}{p_{\text{BRDF}} + p_{\text{lamp}}}$$

- Each way's weight: its share of the two densities.

Veach's balance heuristic

$$w_{\text{BRDF}} = \frac{p_{\text{BRDF}}}{p_{\text{BRDF}} + p_{\text{lamp}}}, \qquad w_{\text{lamp}} = \frac{p_{\text{lamp}}}{p_{\text{BRDF}} + p_{\text{lamp}}}$$

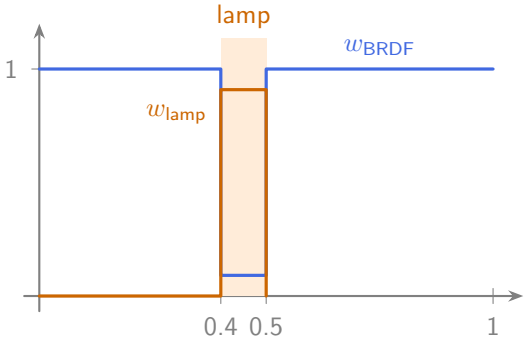


the toy, case 1

- Each way's weight: its share of the two densities.
- On the lamp: $w_{\text{lamp}} = 10/11$, $w_{\text{BRDF}} = 1/11$.

Veach's balance heuristic

$$w_{\text{BRDF}} = \frac{p_{\text{BRDF}}}{p_{\text{BRDF}} + p_{\text{lamp}}}, \qquad w_{\text{lamp}} = \frac{p_{\text{lamp}}}{p_{\text{BRDF}} + p_{\text{lamp}}}$$



the toy, case 1

- Each way's weight: its share of the two densities.
- On the lamp: $w_{\text{lamp}} = 10/11$, $w_{\text{BRDF}} = 1/11$.
- Each way counts most where it picks most often.

It simplifies: divide by the sum of the densities

$$w_{\text{BRDF}} \frac{f}{p_{\text{BRDF}}} = \frac{p_{\text{BRDF}}}{p_{\text{BRDF}} + p_{\text{lamp}}} \cdot \frac{f}{p_{\text{BRDF}}}$$

It simplifies: divide by the sum of the densities

$$w_{\text{BRDF}} \frac{f}{p_{\text{BRDF}}} = \frac{p_{\text{BRDF}}}{p_{\text{BRDF}} + p_{\text{lamp}}} \cdot \frac{f}{p_{\text{BRDF}}}$$

$$= \frac{f}{p_{\text{BRDF}} + p_{\text{lamp}}}$$

It simplifies: divide by the sum of the densities

$$w_{\text{BRDF}} \frac{f}{p_{\text{BRDF}}} = \frac{p_{\text{BRDF}}}{p_{\text{BRDF}} + p_{\text{lamp}}} \cdot \frac{f}{p_{\text{BRDF}}}$$

$$= \frac{f}{p_{\text{BRDF}} + p_{\text{lamp}}}$$

The same for $w_{\text{lamp}} \frac{f}{p_{\text{lamp}}}$. Every pick, either way:

$$\frac{f(\mathbf{l})}{p_{\text{BRDF}}(\mathbf{l}) + p_{\text{lamp}}(\mathbf{l})}$$

It simplifies: divide by the sum of the densities

$$w_{\text{BRDF}} \frac{f}{p_{\text{BRDF}}} = \frac{p_{\text{BRDF}}}{p_{\text{BRDF}} + p_{\text{lamp}}} \cdot \frac{f}{p_{\text{BRDF}}}$$

$$= \frac{f}{p_{\text{BRDF}} + p_{\text{lamp}}}$$

The same for $w_{\text{lamp}} \frac{f}{p_{\text{lamp}}}$. Every pick, either way:

$$\frac{f(\mathbf{l})}{p_{\text{BRDF}}(\mathbf{l}) + p_{\text{lamp}}(\mathbf{l})}$$

- Small only where both ways rarely pick. No way's speck survives.

Worked: two toys, four estimates

Case 2: shiny lobe, $\rho \cos = 20$ on $[0.5, 0.55]$; lamp $L = 2$ on $[0.25, 0.75]$.

	case 1 (exact 1): rough, small lamp	case 2 (exact 2): shiny, big lamp
BRDF picks only		
lamp picks only		
average of one each		
MIS, one each		

Try first: case 2's column.

Worked: two toys, four estimates

Case 2: shiny lobe, $\rho \cos = 20$ on $[0.5, 0.55]$; lamp $L = 2$ on $[0.25, 0.75]$.

	case 1 (exact 1): rough, small lamp	case 2 (exact 2): shiny, big lamp
BRDF picks only	10 (1 in 10) or 0	
lamp picks only	1	
average of one each	5.5 or 0.5	
MIS, one each		

Try first: case 2's column.

Worked: two toys, four estimates

Case 2: shiny lobe, $\rho \cos = 20$ on $[0.5, 0.55]$; lamp $L = 2$ on $[0.25, 0.75]$.

	case 1 (exact 1): rough, small lamp	case 2 (exact 2): shiny, big lamp
BRDF picks only	10 (1 in 10) or 0	2
lamp picks only	1	20 (1 in 10) or 0
average of one each	5.5 or 0.5	11 or 1
MIS, one each		

Try first: case 2's column.

Worked: two toys, four estimates

Case 2: shiny lobe, $\rho \cos = 20$ on $[0.5, 0.55]$; lamp $L = 2$ on $[0.25, 0.75]$.

	case 1 (exact 1): rough, small lamp	case 2 (exact 2): shiny, big lamp
BRDF picks only	10 (1 in 10) or 0	2
lamp picks only	1	20 (1 in 10) or 0
average of one each	5.5 or 0.5	11 or 1
MIS, one each	1.82 or 0.91	

Try first: case 2's column.

Worked: two toys, four estimates

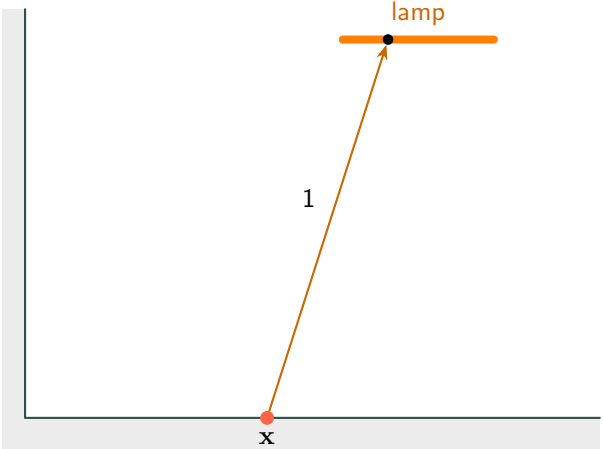
Case 2: shiny lobe, $\rho \cos = 20$ on $[0.5, 0.55]$; lamp $L = 2$ on $[0.25, 0.75]$.

	case 1 (exact 1): rough, small lamp	case 2 (exact 2): shiny, big lamp
BRDF picks only	10 (1 in 10) or 0	2
lamp picks only	1	20 (1 in 10) or 0
average of one each	5.5 or 0.5	11 or 1
MIS, one each	1.82 or 0.91	3.64 or 1.82

Try first: case 2's column.

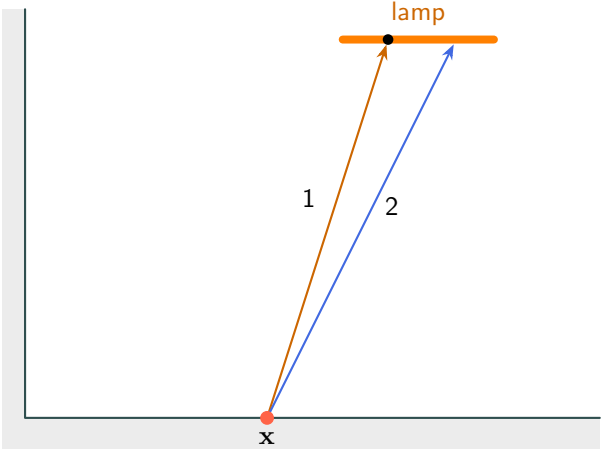
- MIS is never more than twice off, in either case. Each way alone fails once.

Both ways in the path tracer



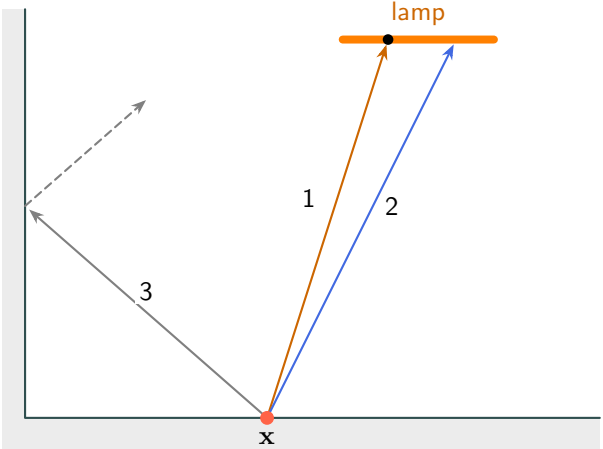
- 1: the shadow ray adds $\frac{f}{p_{\text{lamp}} + p_{\text{BRDF}}}$, with p_{BRDF} of its direction.

Both ways in the path tracer



- 1: the shadow ray adds $\frac{f}{p_{\text{lamp}} + p_{\text{BRDF}}}$, with p_{BRDF} of its direction.
- 2: a bounce onto the lamp: its L_e times w_{BRDF}, with p_{lamp} at the hit.

Both ways in the path tracer



- 1: the shadow ray adds $\frac{f}{p_{\text{lamp}} + p_{\text{BRDF}}}$, with p_{BRDF} of its direction.
- 2: a bounce onto the lamp: its L_e times w_{BRDF}, with p_{lamp} at the hit.
- 3: a bounce onto a wall goes on as in lecture 20.

The weights, in Python

```
def lamp_share(lamp, x, n, material):
    ... as before, up to the shadow ray ...
    p_lamp = r**2 / (lamp.area * cos_y)
    p_brdf = cos_x / math.pi
    return material.albedo / math.pi * lamp.material.emission * cos_x / (p_lamp + p_brdf)
```

The weights, in Python

```
def lamp_share(lamp, x, n, material):
    ... as before, up to the shadow ray ...
    p_lamp = r**2 / (lamp.area * cos_y)
    p_brdf = cos_x / math.pi
    return material.albedo / math.pi * lamp.material.emission * cos_x / (p_lamp + p_brdf)

def trace(ray, depth, p_brdf=None):
    ... the hit, the sky ...
    if isinstance(material, Lambertian):
        ... depth test, x, as in lecture 20 ...
        c = lamp_share(lamp, x, rec.normal, material)
        d = (rec.normal + random_unit_vector()).normalized()
        p = rec.normal.dot(d) / math.pi
        return c + material.albedo * trace(Ray(x, d), depth - 1, p)
```

mirror and glass call `trace(..., depth - 1)` as before: `p_brdf` is None

The weights, in Python

```
def lamp_share(lamp, x, n, material):
    ... as before, up to the shadow ray ...
    p_lamp = r**2 / (lamp.area * cos_y)
    p_brdf = cos_x / math.pi
    return material.albedo / math.pi * lamp.material.emission * cos_x / (p_lamp + p_brdf)

def trace(ray, depth, p_brdf=None):
    ... the hit, the sky ...
    if isinstance(material, Lambertian):
        ... depth test, x, as in lecture 20 ...
        c = lamp_share(lamp, x, rec.normal, material)
        d = (rec.normal + random_unit_vector()).normalized()
        p = rec.normal.dot(d) / math.pi
        return c + material.albedo * trace(Ray(x, d), depth - 1, p)
    elif isinstance(material, DiffuseLight):
        if p_brdf is None:  return material.emission
```

mirror and glass call `trace(..., depth - 1)` as before: `p_brdf` is None

The weights, in Python

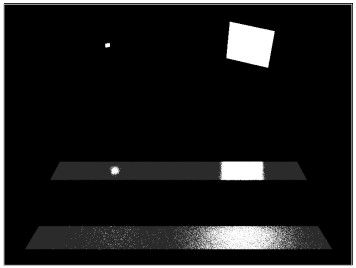
```
def lamp_share(lamp, x, n, material):
    ... as before, up to the shadow ray ...
    p_lamp = r**2 / (lamp.area * cos_y)
    p_brdf = cos_x / math.pi
    return material.albedo / math.pi * lamp.material.emission * cos_x / (p_lamp + p_brdf)

def trace(ray, depth, p_brdf=None):
    ... the hit, the sky ...
    if isinstance(material, Lambertian):
        ... depth test, x, as in lecture 20 ...
        c = lamp_share(lamp, x, rec.normal, material)
        d = (rec.normal + random_unit_vector()).normalized()
        p = rec.normal.dot(d) / math.pi
        return c + material.albedo * trace(Ray(x, d), depth - 1, p)
    elif isinstance(material, DiffuseLight):
        if p_brdf is None:  return material.emission
        p_lamp = rec.t**2 / (lamp.area * -ray.d.dot(lamp.n))
        return p_brdf / (p_brdf + p_lamp) * material.emission
```

mirror and glass call `trace(..., depth - 1)` as before: `p_brdf` is None

Rendered: two plates, two lamps, three ways

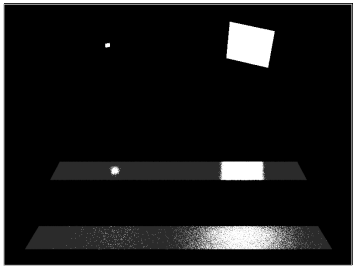
back: shiny plate; front: rough plate; small lamp left, big lamp right



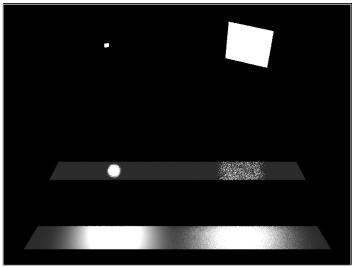
by the BRDF only

Rendered: two plates, two lamps, three ways

back: shiny plate; front: rough plate; small lamp left, big lamp right



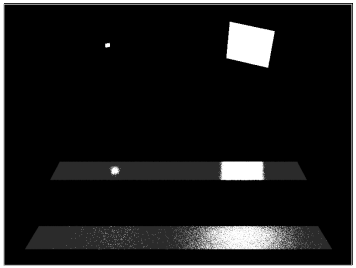
by the BRDF only



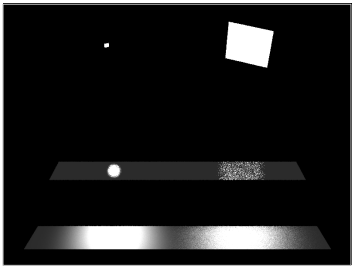
on the lamps only

Rendered: two plates, two lamps, three ways

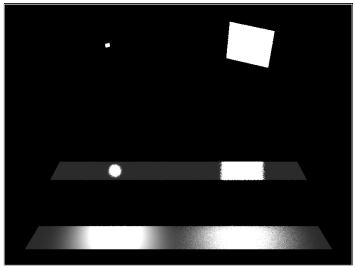
back: shiny plate; front: rough plate; small lamp left, big lamp right



by the BRDF only



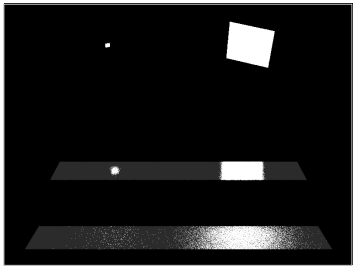
on the lamps only



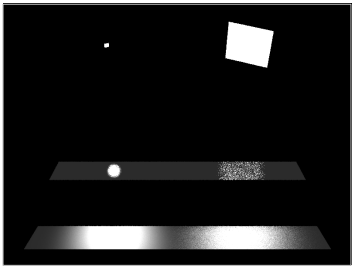
both: MIS

Rendered: two plates, two lamps, three ways

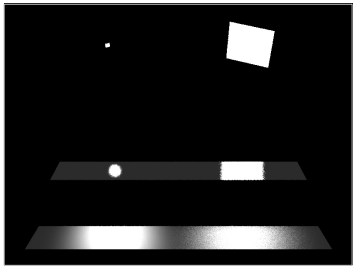
back: shiny plate; front: rough plate; small lamp left, big lamp right



by the BRDF only



on the lamps only

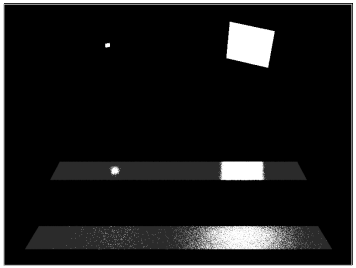


both: MIS

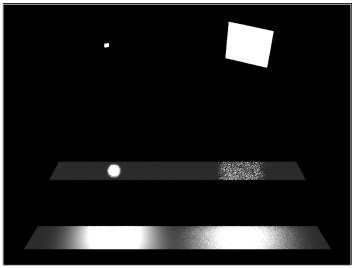
- Each way fails on one corner of the table.

Rendered: two plates, two lamps, three ways

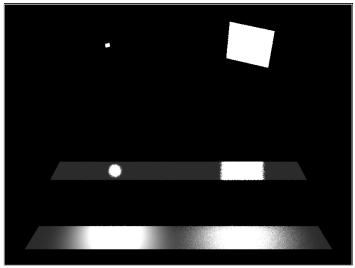
back: shiny plate; front: rough plate; small lamp left, big lamp right



by the BRDF only



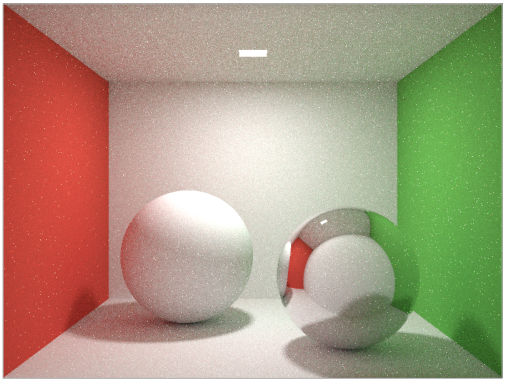
on the lamps only



both: MIS

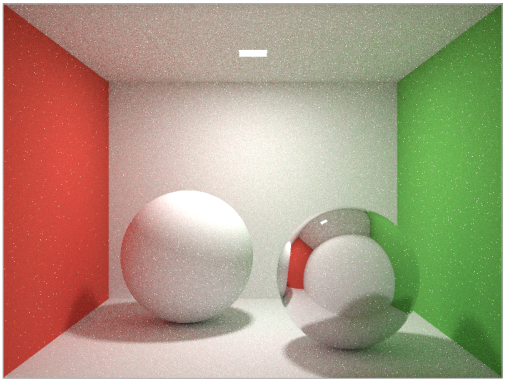
- Each way fails on one corner of the table.
- MIS is clean in all four, with the same rays.

Rendered: the two rooms, with MIS

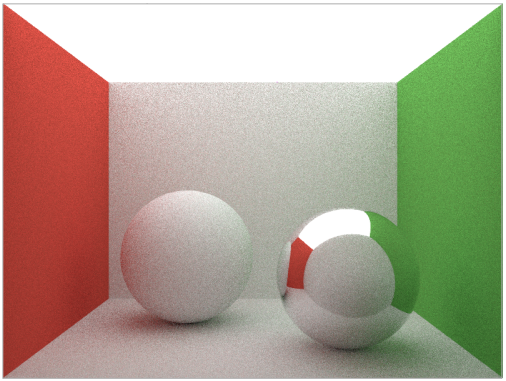


small lamp, MIS, 16 paths

Rendered: the two rooms, with MIS

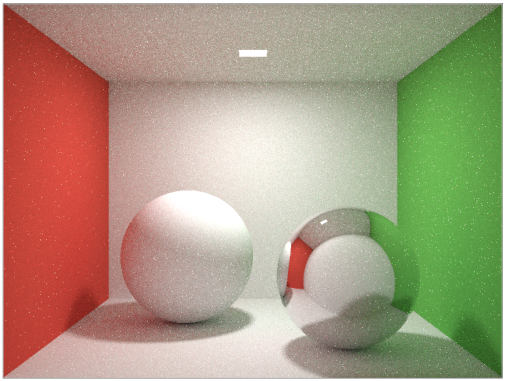


small lamp, MIS, 16 paths

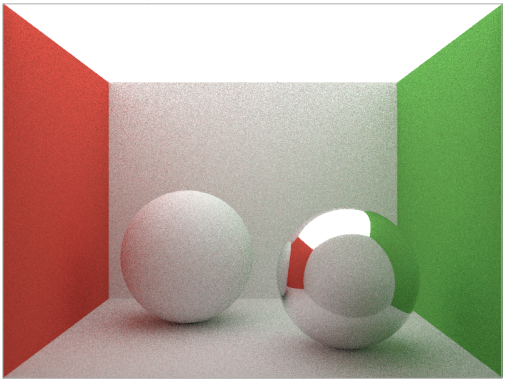


glowing ceiling, MIS, 16 paths

Rendered: the two rooms, with MIS



small lamp, MIS, 16 paths



glowing ceiling, MIS, 16 paths

- One tracer, no switch: each room as clean as its better way.

Quick check 2: the weights at one direction

At a direction $\mathbf{l}$: $p_{\textbf{BRDF}} = 0.3$, $p_{\textbf{lamp}} = 0.9$, $f = 0.6$.

1. The balance weights w_{BRDF} and w_{lamp}?
2. With MIS, what is a pick at $\mathbf{l}$ worth, made either way?
3. Without MIS, a BRDF pick at $\mathbf{l}$? A lamp pick?
4. Why must the weights add up to 1?

- Every pick: f over the sum of the two densities.

Quick check 2: the weights at one direction

At a direction $\mathbf{l}$: $p_{\text{BRDF}} = 0.3$, $p_{\text{lamp}} = 0.9$, $f = 0.6$.

1.	The balance weights w_{BRDF} and w_{lamp}?	0.25, 0.75
2.	With MIS, what is a pick at $\mathbf{l}$ worth, made either way?	$0.6/1.2 = 0.5$
3.	Without MIS, a BRDF pick at $\mathbf{l}$? A lamp pick?	2, 0.67
4.	Why must the weights add up to 1?	right on average

- Every pick: f over the sum of the two densities.

Recap: multiple importance sampling

pick by whichever covers fewer directions

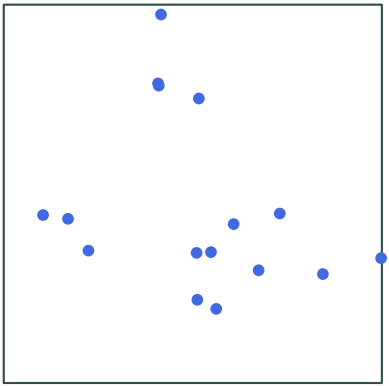
a pick either way: $\dfrac{f}{p_{\text{BRDF}} + p_{\text{lamp}}}$

never the best, never bad

Better random numbers

Stratified and low-discrepancy samples

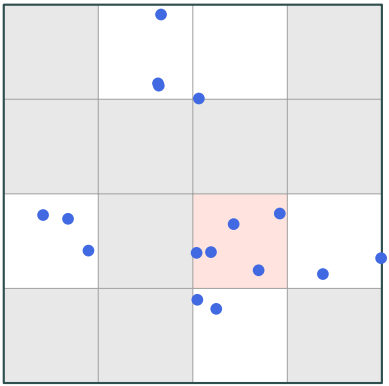
Random picks clump and leave holes



16 random darts

- Independent random picks land anywhere, even on each other.

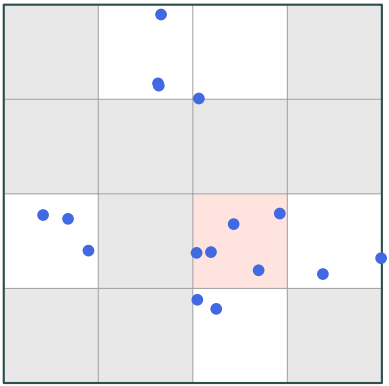
Random picks clump and leave holes



16 random darts

- Independent random picks land anywhere, even on each other.
- Here 10 of 16 cells are empty; one holds 5.

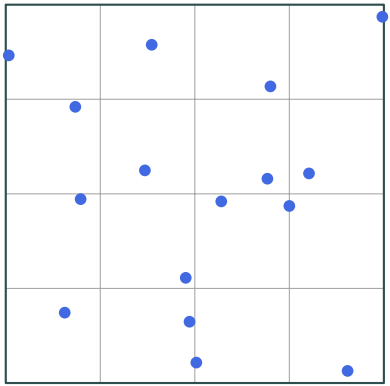
Random picks clump and leave holes



16 random darts

- Independent random picks land anywhere, even on each other.
- Here 10 of 16 cells are empty; one holds 5.
- A clump measures one spot five times; a hole, never.

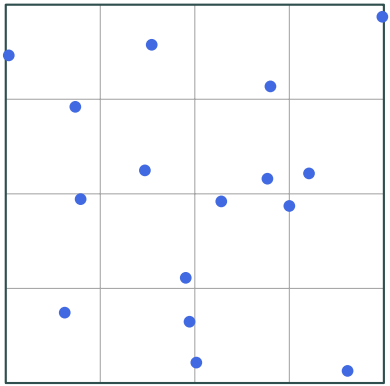
A jittered grid: one random pick per cell



16 jittered darts

- Cut the square into $\sqrt{N} \times \sqrt{N}$ cells.

A jittered grid: one random pick per cell



16 jittered darts

- Cut the square into $\sqrt{N} \times \sqrt{N}$ cells.
- One dart per cell, at a random spot in it:

$$\left(\frac{i + \xi_1}{4}, \frac{j + \xi_2}{4} \right)$$

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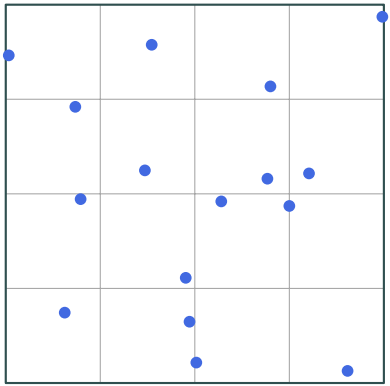

16 jittered darts

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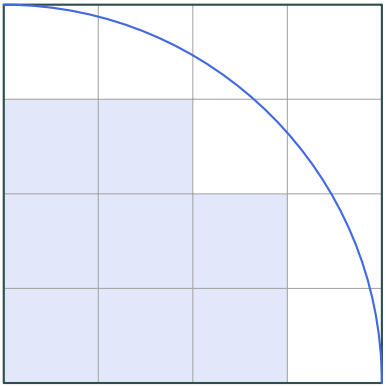
- One dart per cell, at a random spot in it:

$$\left(\frac{i + \xi_1}{4}, \frac{j + \xi_2}{4} \right)$$

- Still right on average. No clumps, no holes.

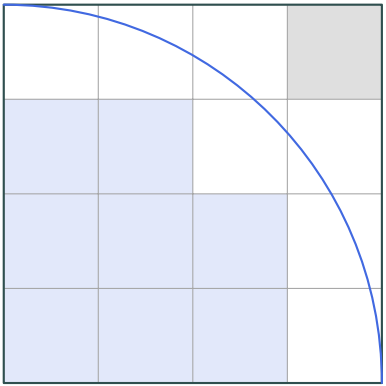
also called stratified sampling: one sample per stratum

Only the cells the edge crosses can be wrong



- 8 cells lie wholly inside: their darts surely count 1.

Only the cells the edge crosses can be wrong



- 8 cells lie wholly inside: their darts surely count 1.
- 1 cell lies wholly outside: surely 0.

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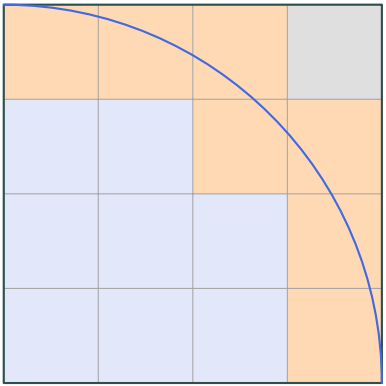
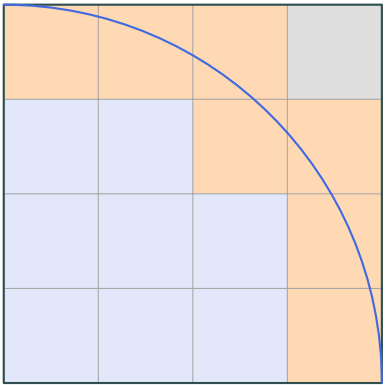


- 8 cells lie wholly inside: their darts surely count 1.
- 1 cell lies wholly outside: surely 0.

- Only the 7 cells the edge crosses are left to chance.

Only the cells the edge crosses can be wrong



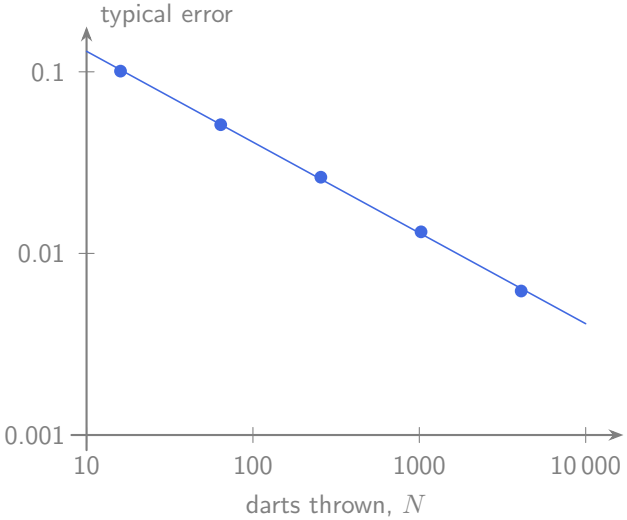
- 8 cells lie wholly inside: their darts surely count 1.
- 1 cell lies wholly outside: surely 0.

- Only the 7 cells the edge crosses are left to chance.

- N cells: about $2\sqrt{N}$ crossed. Error about $N^{-3/4}$.

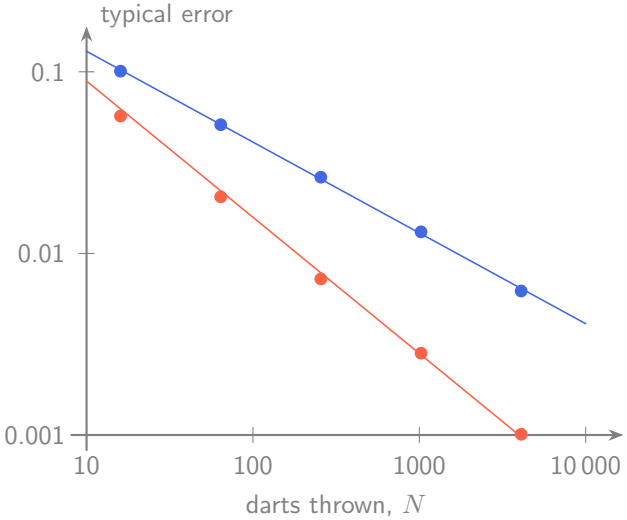
64 cells: 15 crossed; 256 cells: 31

The error falls about as $N^{-3/4}$



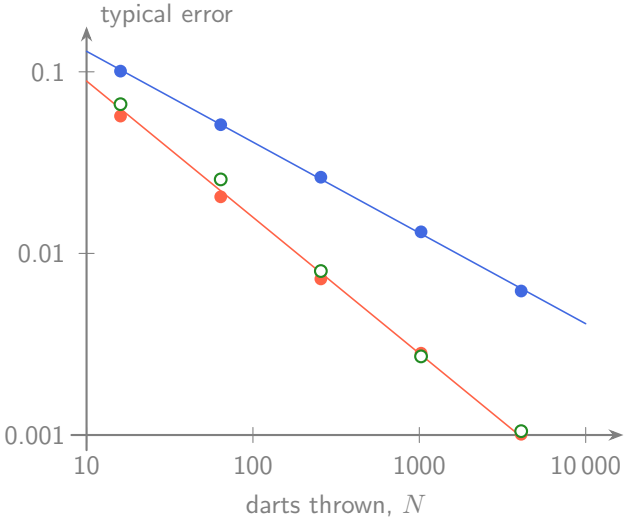
- Random: $0.41/\sqrt{N}$, lecture 20's line.

The error falls about as $N^{-3/4}$



- Random: $0.41/\sqrt{N}$, lecture 20's line.
- Jittered: close to $0.5\,N^{-3/4}$. At 4096 darts, six times smaller.

The error falls about as $N^{-3/4}$



- Random: $0.41/\sqrt{N}$, lecture 20's line.
- Jittered: close to $0.5\,N^{-3/4}$. At 4096 darts, six times smaller.
- Halton points (next): about the same.

measured: 400 runs at each N

Halton: mirror the digits after the point

k	base 2	mirrored	value
1	1		
2	10		
3	11		
4	100		
5	101		
6	110		
7	111		
8	1000		

- Write k in base 2, mirror its digits after the point.

Halton: mirror the digits after the point

k	base 2	mirrored	value
1	1	0.1	0.5
2	10	0.01	0.25
3	11	0.11	0.75
4	100	0.001	0.125
5	101	0.101	0.625
6	110	0.011	0.375
7	111	0.111	0.875
8	1000	0.0001	0.0625

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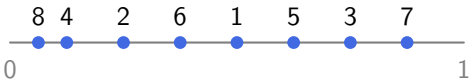
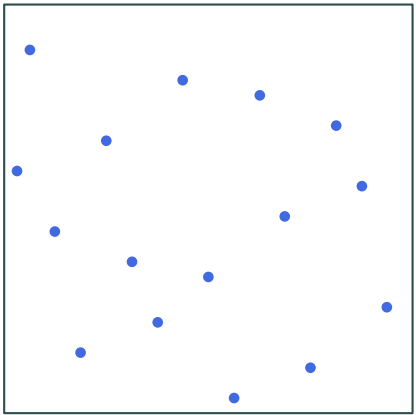


- Each new number falls in the largest gap left.

the radical inverse of k (van der Corput)

Halton points fill the square at every N

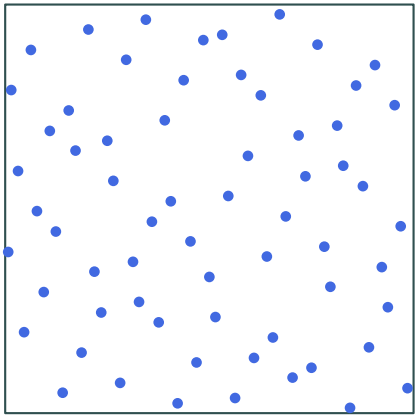
the first 16



Halton: x in base 2, y in base 3

Halton points fill the square at every N

the first 64



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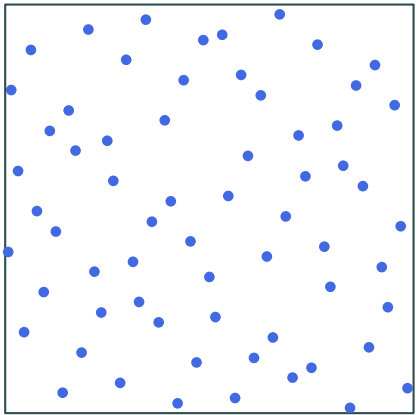

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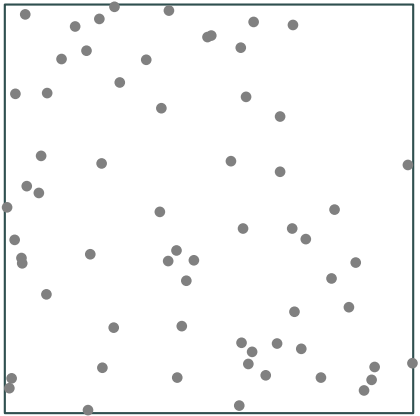
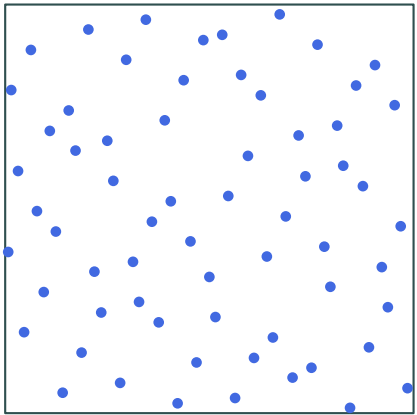


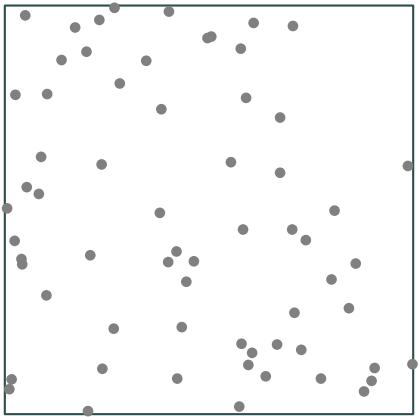
64 random points

Halton points fill the square at every N

the first 64



Halton: x in base 2, y in base 3



64 random points

no grid,
no clumps

The radical inverse, in Python

```
def radical_inverse(k, base):
    f, r = 1.0, 0.0
    while k > 0:
        f = f / base
        r = r + f * (k % base)
        k = k // base
    return r
```

`k % base`: the last digit; `k // base`: the rest; `f`: the place of the next digit after the point

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```

$k = 6$, base 2:

f	digit	r	k
0.5	0	0	3
0.25	1	0.25	1
0.125	1	0.375	0

returns $0.375 = 0.011_2$

`k % base`: the last digit; `k // base`: the rest; `f`: the place of the next digit after the point

The radical inverse, in Python

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0.25	1	0.25	1
0.125	1	0.375	0

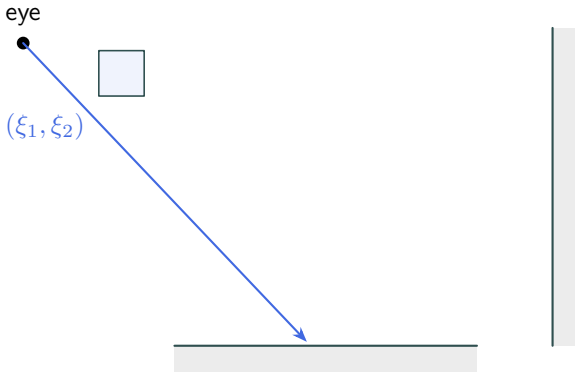
returns $0.375 = 0.011_2$

```
sx, sy = random.random(), random.random()  # once per pixel
x = (radical_inverse(k, 2) + sx) % 1.0
y = (radical_inverse(k, 3) + sy) % 1.0
```

- A random shift per pixel, wrapped round: random again, still even.

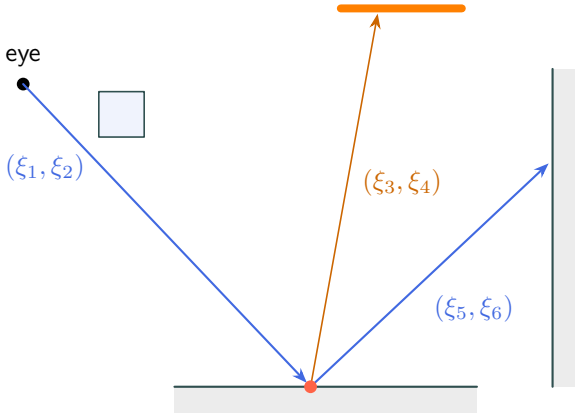
`k % base`: the last digit; `k // base`: the rest; `f`: the place of the next digit after the point

Which random numbers a path uses



pixel spot, then at each chalk hit:
a point on the lamp, a bounce

Which random numbers a path uses



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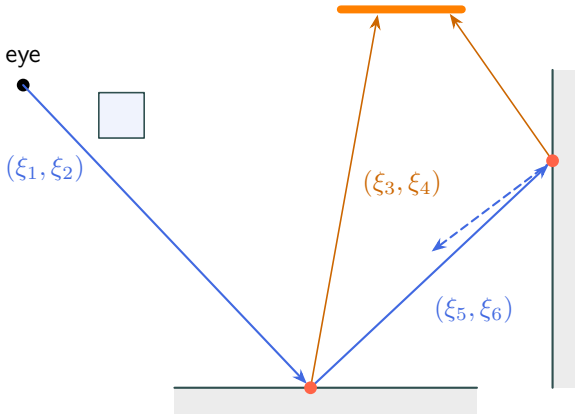
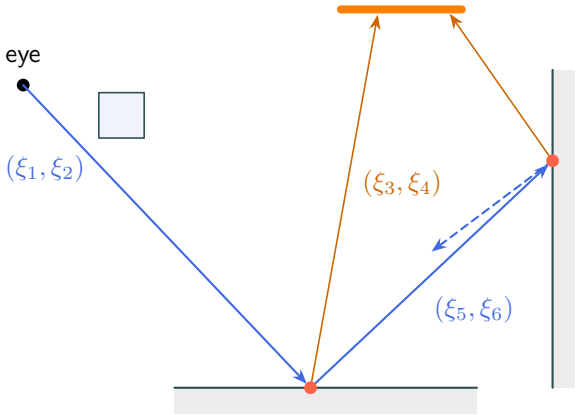


pixel spot, then at each chalk hit:
a point on the lamp, a bounce

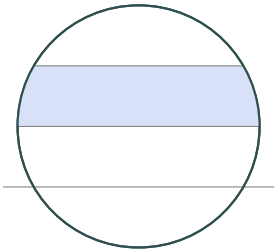
- Jitter each pair on its own.

Which random numbers a path uses



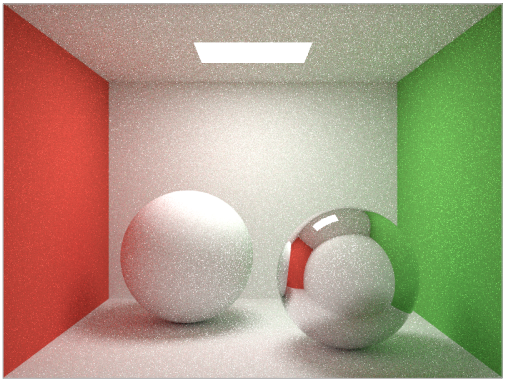
pixel spot, then at each chalk hit:
a point on the lamp, a bounce

- Jitter each pair on its own.
- Even cells stay even on the sphere.



even heights z:
equal areas (Archimedes)

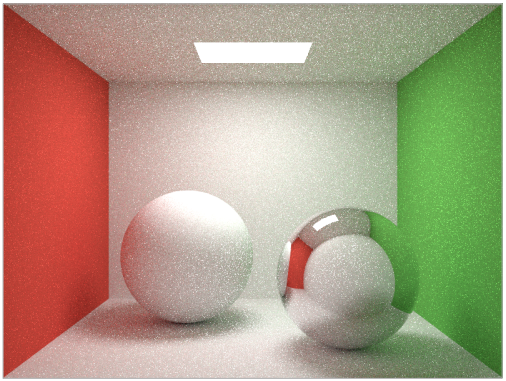
Rendered: 16 paths, random and jittered



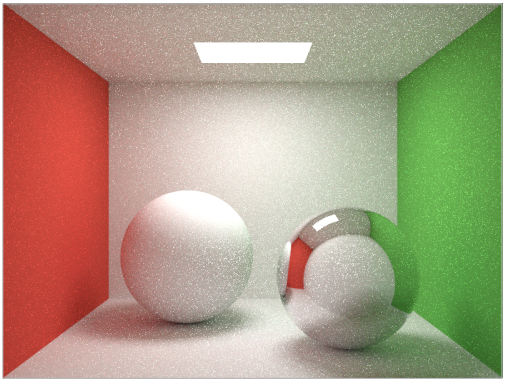
random numbers, 16 paths

- Same room, same work.

Rendered: 16 paths, random and jittered



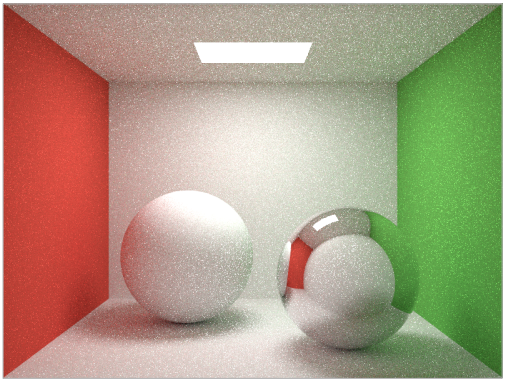
random numbers, 16 paths



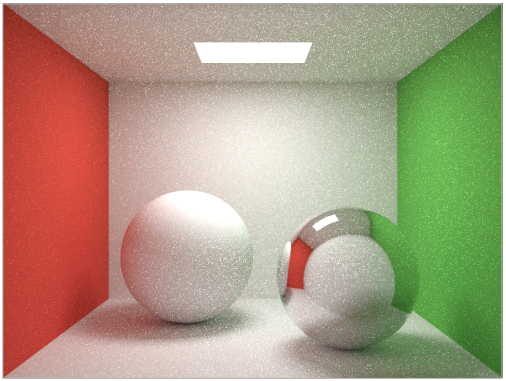
jittered pairs, 16 paths

- Same room, same work.

Rendered: 16 paths, random and jittered



random numbers, 16 paths



jittered pairs, 16 paths

- Same room, same work.
- Edges and soft shadows calm down; deep bounces less.

Recap: better random numbers

jitter: one pick per cell; error about $N^{-3/4}$ in 2D

Halton: mirrored digits, any N, a random shift per pixel

in a path: jitter each pair on its own

Summary

1 Two ways to find the lamp

by the BRDF: rare hits on small lamps

on the lamp: wild near big lamps

2 Area lamps

$$p_{\text{lamp}} = \frac{r^2}{A \cos \theta_y}$$

one pick: $\rho \, L_e \, A \cos \theta \cos \theta_y / r^2$

3 Multiple importance sampling

every pick: $\dfrac{f}{p_{\text{BRDF}} + p_{\text{lamp}}}$

Veach's balance heuristic

4 Better random numbers

jitter, Halton: error about $N^{-3/4}$

a random shift per pixel

jitter each pair of a path on its own

Reading

Pharr, Jakob and Humphreys, *Physically Based Rendering*, 4th edition (2023; free at pbr-book.org)

chapter 2: Monte Carlo chapter 8: stratified and Halton samples

chapter 12: area lights chapter 13: the path tracer, with MIS

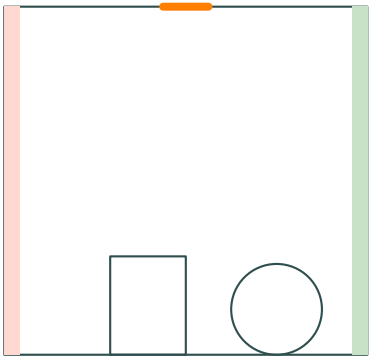
Veach, *Robust Monte Carlo Methods for Light Transport Simulation*, PhD thesis, Stanford, 1997

chapter 9: multiple importance sampling, the balance heuristic, the glossy plates

Shirley, *Ray Tracing: The Rest of Your Life* (book 3 of the RTIOW series)

densities, mixtures of a lamp and a cosine: the gentlest way in

Next: the extra-credit assignment, then Extra 2



extra credit: your tracer,
a small lamp, MIS

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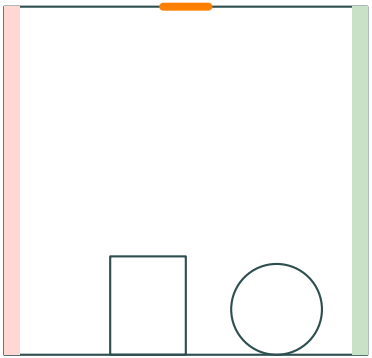

extra credit: your tracer,
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Extra 2: a surface of tiny mirrors

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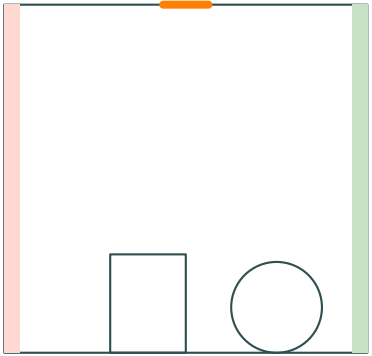

extra credit: your tracer,
a small lamp, MIS

- Real shiny materials, sampled by today's rules.



Extra 2: a surface of tiny mirrors