Lecture 21: Recursion and the Call Stack

Comp 102

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Forman Christian University

Stack Overflow

A Website Every Programmer Knows

- Stuck on a bug? Programmers ask on a website.



A Website Every Programmer Knows

- Stuck on a bug? Programmers ask on a website.
- Its name: **Stack Overflow**.



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A Website Every Programmer Knows

- Stuck on a bug? Programmers ask on a website.
- Its name: **Stack Overflow**.
- Millions of programmers use it every day.



Why that name? What is a **stack**, and how does it **overflow**?

Write a guess on your paper (30 s). We answer it today.

Pictures Inside Pictures



Two mirrors face each other.

Pictures Inside Pictures



Two mirrors face each other.



A cocoa tin (1904) with a picture of the **same tin** on it.

Pictures Inside Pictures



Two mirrors face each other.



A cocoa tin (1904) with a picture of the **same tin** on it.

Each picture holds a **smaller copy of itself**.

Today

1. A function that **calls itself**.

Today

1. A function that **calls itself**.
2. The **stack** of frames Python keeps for every call.

Today

1. A function that **calls itself**.
2. The **stack** of frames Python keeps for every call.
3. The day the frames **run out**... and the name of the website.

Folders Inside Folders

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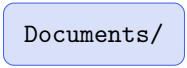
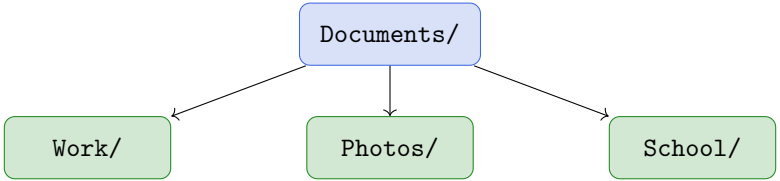


Folders Inside Folders



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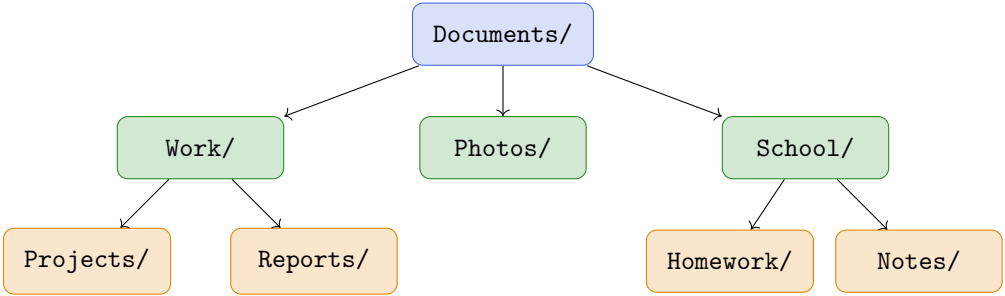

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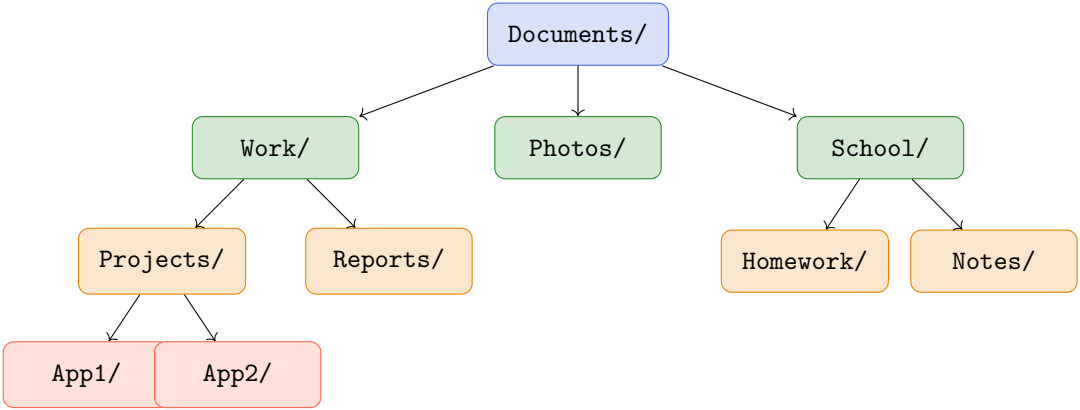
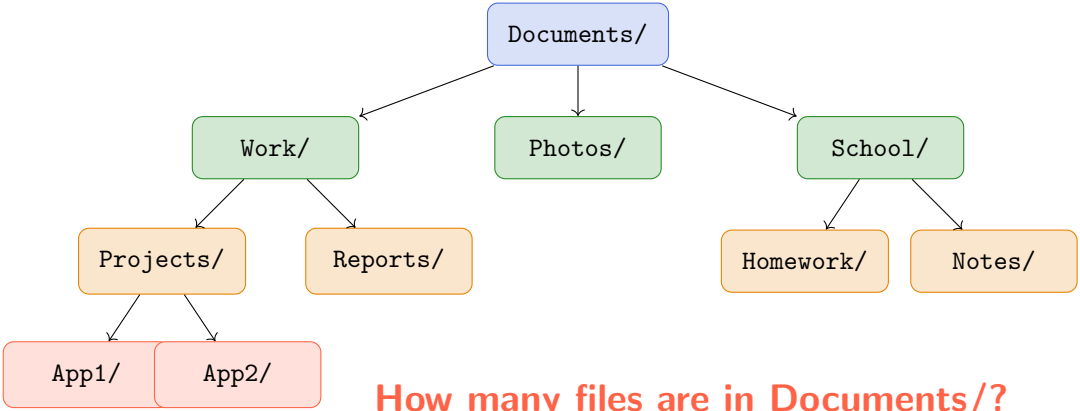


Folders Inside Folders



How many files are in Documents/?

We do not know how deep the folders go.

Counting the Files

To count the files in Documents/:

Counting the Files

To count the files in Documents/:

1. Count the files right here (say, 3).

Counting the Files

To count the files in Documents/:

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2. Count the files in Work/ ← same job, smaller!

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1. Count the files right here (say, 3).
2. Count the files in Work/ ← same job, smaller!
3. Count the files in Photos/ ← same job, smaller!

Counting the Files

To count the files in Documents/:

1. Count the files right here (say, 3).
2. Count the files in Work/ ← same job, smaller!
3. Count the files in Photos/ ← same job, smaller!
4. Count the files in School/ ← same job, smaller!

Counting the Files

To count the files in Documents/:

1. Count the files right here (say, 3).
2. Count the files in Work/ ← same job, smaller!
3. Count the files in Photos/ ← same job, smaller!
4. Count the files in School/ ← same job, smaller!
5. Add them all up.

Counting the Files

To count the files in Documents/:

1. Count the files right here (say, 3).
2. Count the files in Work/ ← same job, smaller!
3. Count the files in Photos/ ← same job, smaller!
4. Count the files in School/ ← same job, smaller!
5. Add them all up.

Stop: a folder with no folders inside. Just count its files.

Counting the Files

To count the files in Documents/:

1. Count the files right here (say, 3).
2. Count the files in Work/ ← same job, smaller!
3. Count the files in Photos/ ← same job, smaller!
4. Count the files in School/ ← same job, smaller!
5. Add them all up.

Stop: a folder with no folders inside. Just count its files.

The job contains a smaller copy of the same job.

What is Recursion?

Recursion: a function **calls itself**

to do a **smaller copy** of the same job.

What is Recursion?

Recursion: a function **calls itself**

to do a **smaller copy** of the same job.

Like the cocoa tin: the picture holds a smaller copy of itself.

Two Parts, Always

1. BASE CASE

When to **STOP**

(the answer is easy)

Two Parts, Always

1. BASE CASE
When to **STOP**
(the answer is easy)

+

2. RECURSIVE CASE
Call yourself on a
SMALLER job

Two Parts, Always

1. BASE CASE
When to **STOP**
(the answer is easy)

+

2. RECURSIVE CASE
Call yourself on a
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Every recursive function needs **both**.

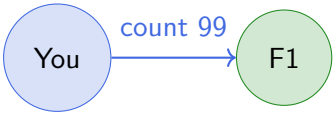
The Lazy Helpers

"Count these 100 sheep for me!"



The Lazy Helpers

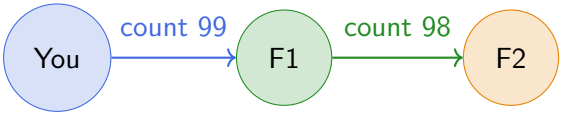
"Count these 100 sheep for me!"



takes 1 sheep

The Lazy Helpers

"Count these 100 sheep for me!"



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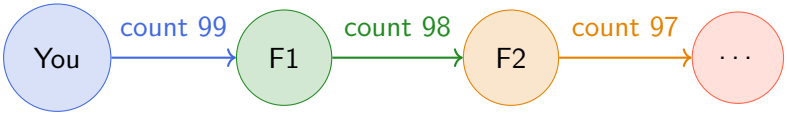

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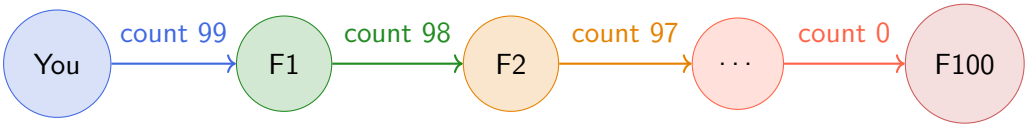

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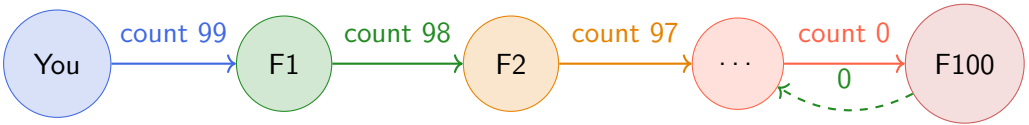

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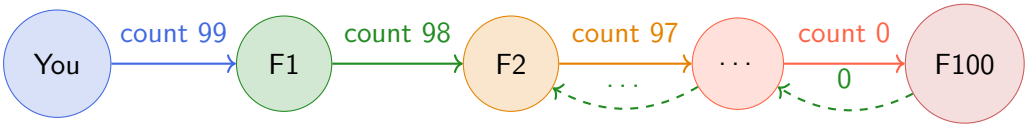

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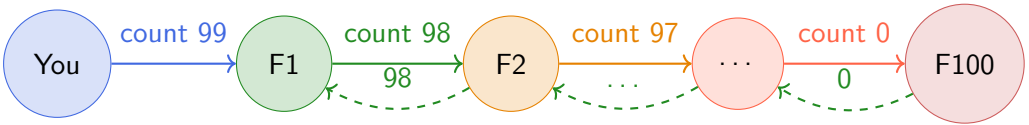

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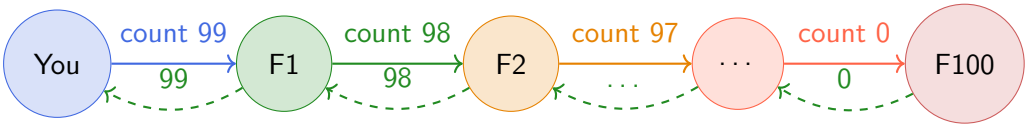

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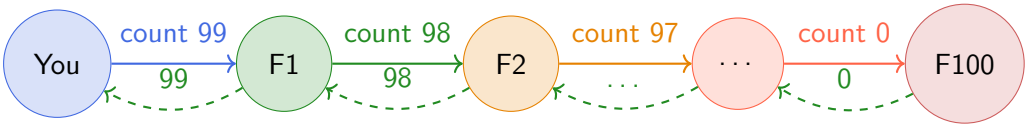

The answers come back up: 100!

First Example: Countdown

```
1  def countdown(n):
2      # Base case
3      if n <= 0:
4          print("Blastoff!")
5          return
6      # Recursive case
7      print(n)
8      countdown(n - 1)
9
10 countdown(5)
```

Output:

```
5
4
3
2
1
Blastoff!
```

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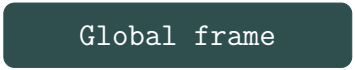
2
1
Blastoff!
```

The Call Stack

Every Call Gets a Frame

- Lecture 10: every function call makes a **new frame**.



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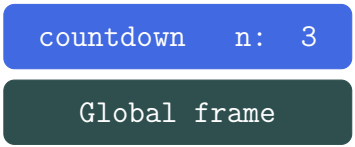
```
countdown(3)
```



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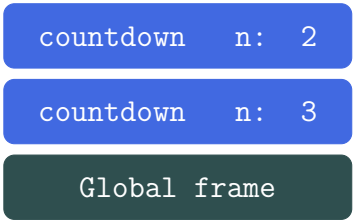

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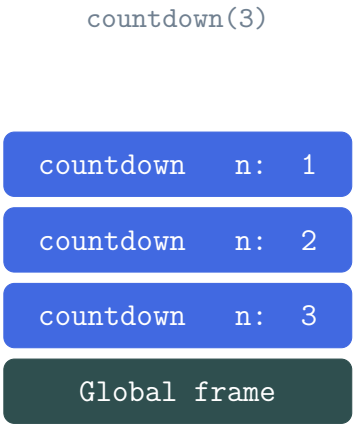

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Every Call Gets a Frame

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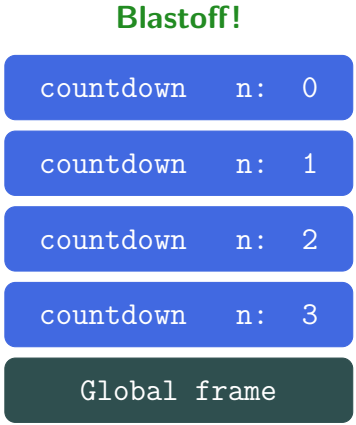
- Each frame has its **own** `n`.



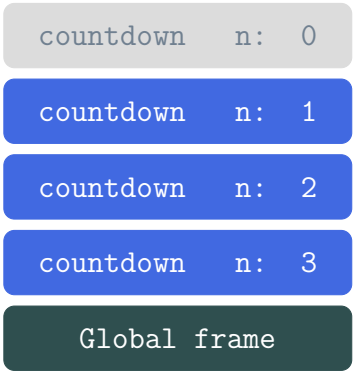
Every Call Gets a Frame

- Lecture 10: every function call makes a **new frame**.
- Each frame has its **own** `n`.
- The newest frame is on **top**.



Every Call Gets a Frame

- Lecture 10: every function call makes a **new frame**.
- Each frame has its **own** `n`.
- The newest frame is on **top**.
- When a call ends, its frame is **removed**.



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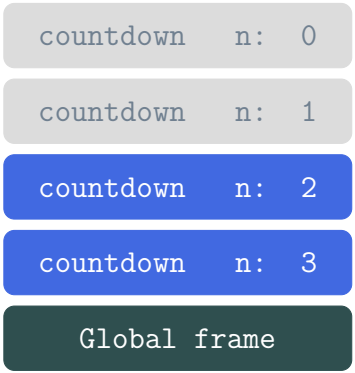

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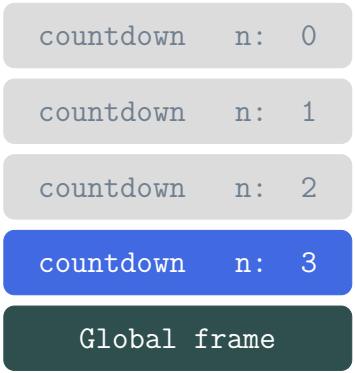
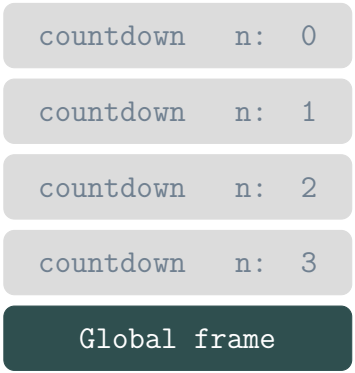


Every Call Gets a Frame

- Lecture 10: every function call makes a **new frame**.
- Each frame has its **own** `n`.
- The newest frame is on **top**.
- When a call ends, its frame is **removed**.

A pile like this is a **stack**: the last one on is the first one off, like plates.



Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```



Global frame

Python works out:

`result = sum_to(4)` ⇒ first call `sum_to(4)`

Python starts on line 8. First it must call `sum_to(4)`.

Tracing sum_to(4): Going Down

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1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 4

Global frame

line 8: result = ?

Python works out:

sum_to(4) ⇒ n = 4

A new frame for this call. In it, n is 4.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 4

Global frame

line 8: result = ?

Python works out:

4 <= 0 ⇒ False

Is 4 <= 0? **No.** So Python skips line 4.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
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8  result = sum_to(4)
```



sum_to n: 4 | 4 + sum_to(3)

Global frame | line 8: result = ?

Python works out:

`n + sum_to(n - 1)` ⇒ `4 + sum_to(3)`

Line 6. To finish it, Python needs `sum_to(3)` first.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
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7
8  result = sum_to(4)
```

sum_to n: 3

sum_to n: 4 — line 6: 4 + ?

Global frame — line 8: result = ?

Python works out:

sum_to(3) ⇒ n = 3

sum_to(4) waits. A new frame: n is 3.

Tracing `sum_to(4)`: Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 3	
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

3 <= 0 ⇒ False

Is 3 <= 0? **No.** So Python skips line 4.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 3	3 + sum_to(2)
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`n + sum_to(n - 1)` ⇒ `3 + sum_to(2)`

Line 6. To finish it, Python needs `sum_to(2)` first.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 2	
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`sum_to(2)` ⇒ `n = 2`

`sum_to(3)` waits. A new frame: `n` is 2.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 2	
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

2 <= 0 ⇒ False

Is 2 <= 0? **No.** So Python skips line 4.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 2	2 + sum_to(1)
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`n + sum_to(n - 1)` ⇒ `2 + sum_to(1)`

Line 6. To finish it, Python needs `sum_to(1)` first.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
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5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 1	
sum_to n: 2	line 6: 2 + ?
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`sum_to(1)` ⇒ `n = 1`

`sum_to(2)` waits. A new frame: `n` is 1.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 1	
sum_to n: 2	line 6: 2 + ?
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`1 <= 0` ⇒ `False`

Is 1 <= 0? **No.** So Python skips line 4.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
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8  result = sum_to(4)
```

sum_to n: 1	1 + sum_to(0)
sum_to n: 2	line 6: 2 + ?
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`n + sum_to(n - 1)` ⇒ `1 + sum_to(0)`

Line 6. To finish it, Python needs `sum_to(0)` first.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
2      # Base case
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5      # Recursive case
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8  result = sum_to(4)
```

sum_to n: 0	
sum_to n: 1	line 6: 1 + ?
sum_to n: 2	line 6: 2 + ?
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`sum_to(0)` ⇒ `n = 0`

`sum_to(1)` waits. A new frame: `n` is 0.

Tracing `sum_to(4)`: Going Down

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 0	
sum_to n: 1	line 6: 1 + ?
sum_to n: 2	line 6: 2 + ?
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`0 <= 0` ⇒ `True`

Is 0 <= 0? **Yes!** This is the base case.

Tracing sum_to(4): Going Down

```
1  def sum_to(n):
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3      if n <= 0:
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7
8  result = sum_to(4)
```

sum_to n: 0	returns 0
sum_to n: 1	line 6: 1 + ?
sum_to n: 2	line 6: 2 + ?
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`return 0` ⇒ 0

sum_to(0) returns 0. No new call. Now we go back up.

Tracing `sum_to(4)`: Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 0	returns 0
sum_to n: 1	line 6: 1 + ?
sum_to n: 2	line 6: 2 + ?
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`return 0` ⇒ 0

Where we stopped: `sum_to(0)` returns 0.

Tracing `sum_to(4)`: Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
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8  result = sum_to(4)
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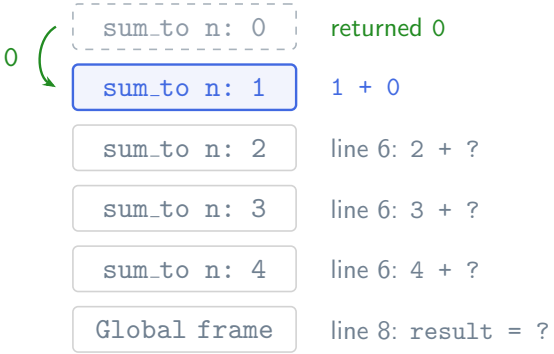

Python works out:

`1 + sum_to(0)` ⇒ `1 + 0`

Its frame is removed. Back in `n`: 1, on line 6. The **?** becomes 0.

Tracing sum_to(4): Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
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5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 1	returns 1
sum_to n: 2	line 6: 2 + ?
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

1 + 0 ⇒ 1

1 + 0 = 1. So sum_to(1) returns 1.

Tracing `sum_to(4)`: Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```





Python works out:

`2 + sum_to(1)` ⇒ `2 + 1`

Its frame is removed. Back in `n: 2`, on line 6. The **?** becomes 1.

Tracing sum_to(4): Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 2	returns 3
sum_to n: 3	line 6: 3 + ?
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

2 + 1 ⇒ 3

2 + 1 = 3. So sum_to(2) returns 3.

Tracing sum_to(4): Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

3

sum_to n: 2	returned 3
sum_to n: 3	3 + 3
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

`3 + sum_to(2)` ⇒ `3 + 3`

Its frame is removed. Back in `n`: 3, on line 6. The **?** becomes 3.

Tracing `sum_to(4)`: Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```

sum_to n: 3	returns 6
sum_to n: 4	line 6: 4 + ?
Global frame	line 8: result = ?

Python works out:

3 + 3 ⇒ 6

3 + 3 = 6. So sum_to(3) returns 6.

Tracing `sum_to(4)`: Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```





Python works out:

`4 + sum_to(3)` ⇒ `4 + 6`

Its frame is removed. Back in `n: 4`, on line 6. The **?** becomes 6.

Tracing sum_to(4): Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```



sum_to n: 4 returns 10

Global frame line 8: result = ?

Python works out:

4 + 6 ⇒ 10

4 + 6 = 10. So sum_to(4) returns 10.

Tracing `sum_to(4)`: Coming Back Up

```
1  def sum_to(n):
2      # Base case
3      if n <= 0:
4          return 0
5      # Recursive case
6      return n + sum_to(n - 1)
7
8  result = sum_to(4)
```




Python works out:

`result = sum_to(4)` ⇒ `result = 10`

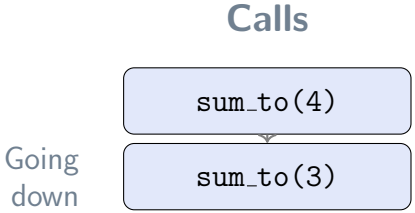
Its frame is removed. Back on line 8: `result` is **10**.

Going Down, Coming Back Up

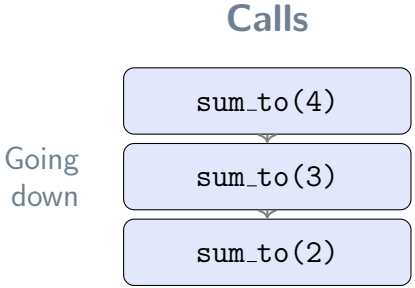
Calls

```
sum_to(4)
```

Going Down, Coming Back Up



Going Down, Coming Back Up



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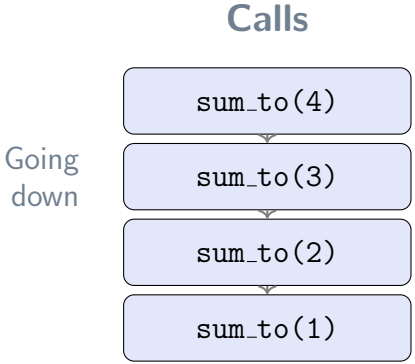

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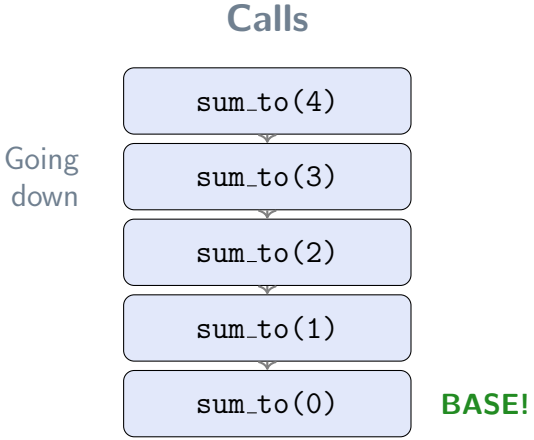

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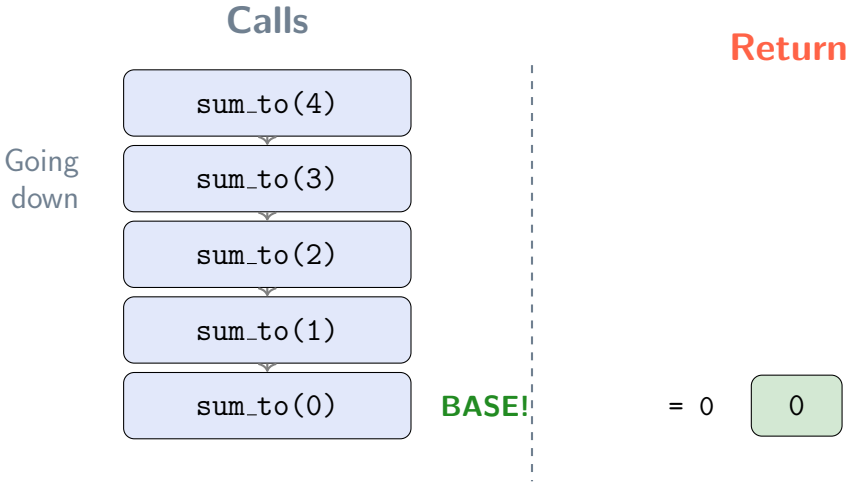
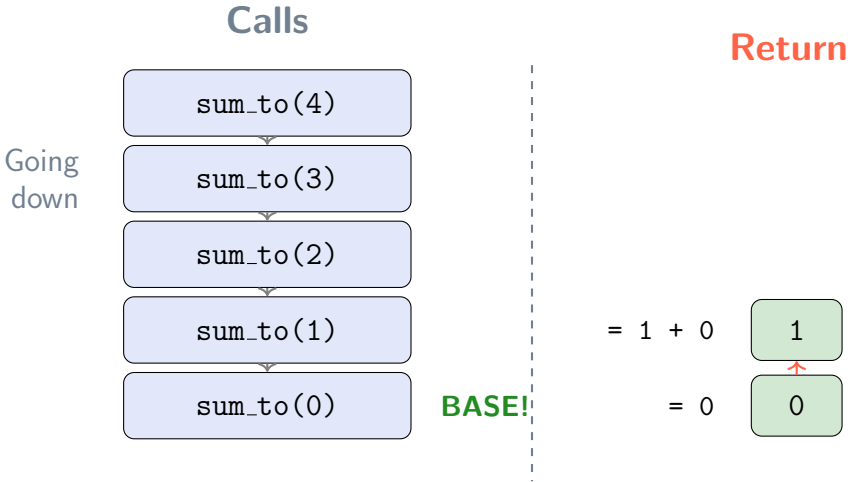


Going Down, Coming Back Up



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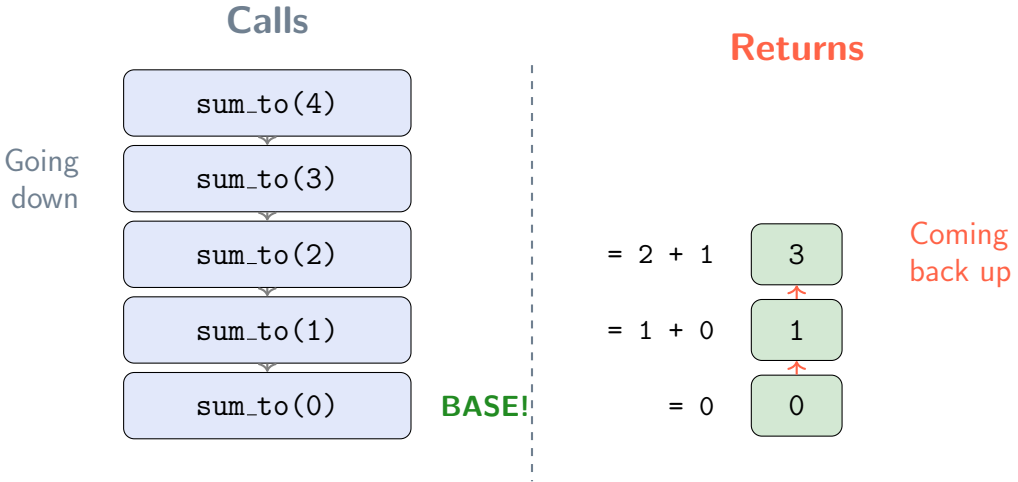

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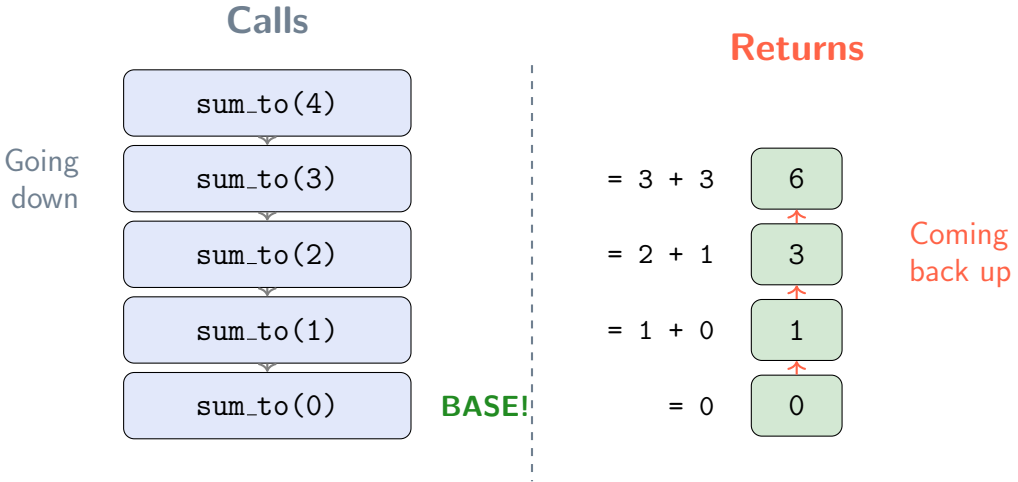

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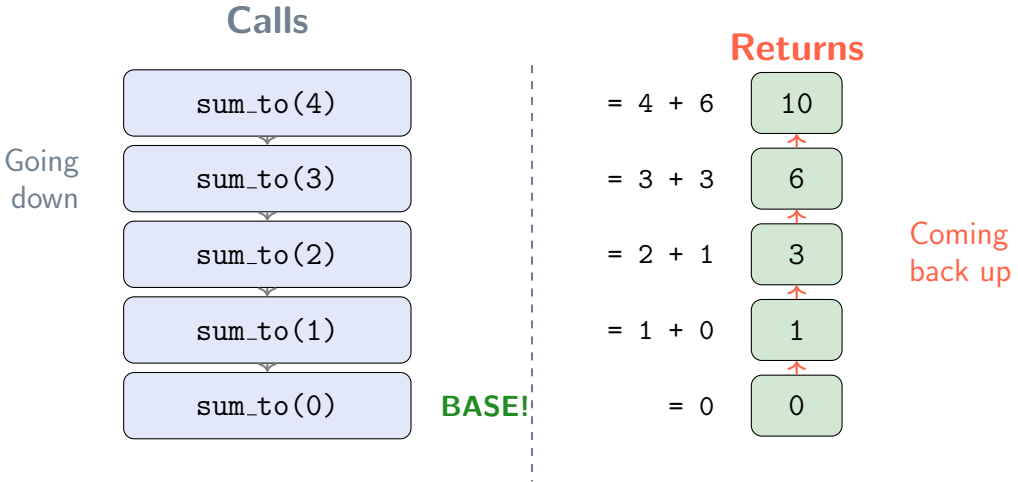

Why `n <= 0` and not `n == 1`?

With `if n <= 0:` the function also handles:

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- `sum_to(0)` → **0** ✓

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With `if n == 1:`, try `sum_to(0)`. It goes $-1, -2, -3, \ldots$

It never reaches the base case.

Why `n <= 0` and not `n == 1`?

With `if n <= 0:` the function also handles:

- `sum_to(0)` → **0** ✓
- `sum_to(-5)` → **0** ✓

With `if n == 1:`, try `sum_to(0)`. It goes $-1, -2, -3, \ldots$

It never reaches the base case.

Always ask: which odd inputs could reach my function?

You Try! *(write the output, 1 min)*

The `print` is now **after** the call. What does `countup(3)` print?

```
1 def countup(n):
2     if n == 0:
3         return
4     countup(n - 1)
5     print(n)
6
7 countup(3)
```

You Try! *(write the output, 1 min)*

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```
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7 countup(3)
```

1
2
3

You Try! *(write the output, 1 min)*

The `print` is now **after** the call. What does `countup(3)` print?

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1 def countup(n):
2     if n == 0:
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7 countup(3)
```

```
1
2
3
```

Why? `countup(3)` must wait for `countup(2)` to finish before it prints 3. The **top** frame prints **first**.

Watch the Frames in Thonny

1. Open `lec21.py`.

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Watch the Frames in Thonny

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2. **Debug current script** (the bug button, or Ctrl+F5).
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4. Each call opens a **new window**: that window is a frame.

At home you can also try `https://pythontutor.com`.

The Day the Frames Run Out

No Base Case

```
1 def infinite(n):
2     print(n)
3     infinite(n - 1)
4
5 infinite(5)
```

No Base Case

```
1 def infinite(n):
2     print(n)
3     infinite(n - 1)
4
5 infinite(5)
```

Output:

```
5
4
...
-993
RecursionError
```

infinite(5)

No Base Case

```
1 def infinite(n):
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```
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...
-993
RecursionError
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infinite(4)
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No Base Case

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4
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```

Output:

```
5
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...
-993
RecursionError
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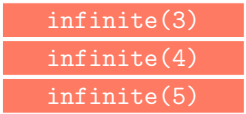

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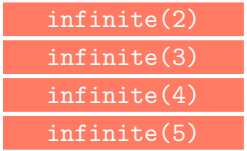

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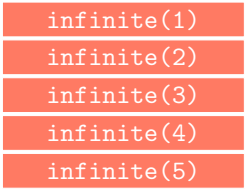

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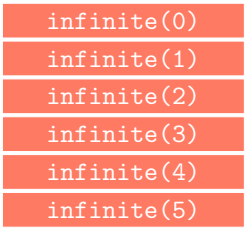

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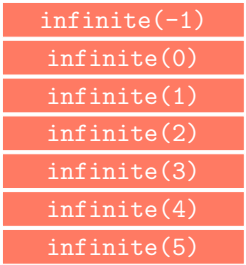

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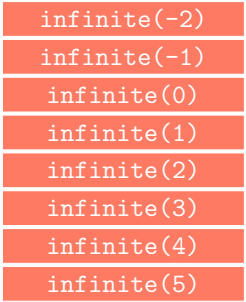

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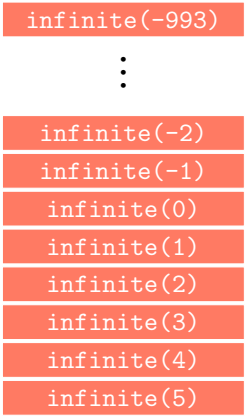

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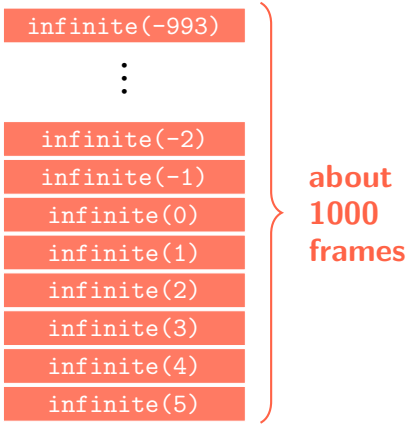

The Pile Has a Limit

- Every frame takes some **memory**.

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- Python allows about **1000** frames on the stack.

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```
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A stack of frames that grows past its limit
is a **stack overflow**.

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A stack of frames that grows past its limit
is a **stack overflow**.

Programmers named their website after the bug they **all** meet.

Bug Hunt! *(write the line number and the fix, 30 s)*

`power(b, e)` should give b^e. For example, `power(2, 3)` is 8.

```
1 def power(b, e):
2     if e == 0:
3         return 1
4     return b * power(b, e)
5
6 print(power(2, 3))
```

Symptom:

```
RecursionError: maximum
recursion depth exceeded
```

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5
6 print(power(2, 3))
```

Symptom:

```
RecursionError: maximum
recursion depth exceeded
```

Line 4. `e` never gets smaller, so `e == 0` never comes.

Fix: `power(b, e - 1)`

Bug Hunt! *(write the line number and the fix, 1 min)*

Count down in **twos**, then say `Blastoff!`

```
1  def twos(n):
2      if n == 0:
3          print("Blastoff!")
4          return
5      print(n)
6      twos(n - 2)
```

Symptom:

`twos(6)` prints `6 4 2 Blastoff!`
`twos(7)` prints `7 5 3 1 -1 -3...`
then `RecursionError`

Bug Hunt! *(write the line number and the fix, 1 min)*

Count down in **twos**, then say `Blastoff!`

```
1  def twos(n):
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```

Symptom:

`twos(6)` prints `6 4 2 Blastoff!`
`twos(7)` prints `7 5 3 1 -1 -3...`
then `RecursionError`

Line 2. From 7, `n` goes $1, -1$: it **jumps over** 0.
Fix: `if n <= 0:`

Bug Hunt! *(write the line number and the fix, 1 min)*

```
1 def sum_to(n):
2     if n <= 0:
3         return 0
4     n + sum_to(n - 1)
5
6 print(sum_to(3))
```

Symptom:

```
TypeError: unsupported
operand type(s) for +:
'int' and 'NoneType'
```

Bug Hunt! *(write the line number and the fix, 1 min)*

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Line 4. It works out `n + ...` and throws it away. No `return`, so the call gives `None`.

Fix: `return n + sum_to(n - 1)`

Bug Hunt! *(write the line number and the fix, 1 min)*

```
1 def sum_to(n):
2     if n <= 0:
3         return 0
4     n + sum_to(n - 1)
5
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```
TypeError: unsupported
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'int' and 'NoneType'
```

Line 4. It works out `n + ...` and throws it away. No `return`, so the call gives `None`.

Fix: `return n + sum_to(n - 1)`
`sum_to(1)` gives `None`. Then `sum_to(2)` tries `2 + None`: **TypeError**.

The Big Three Recursion Bugs

Bug	Symptom	Fix

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Not getting smaller	`RecursionError`	`e - 1`, not `e`

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Missing `return`	`None` or `TypeError`	`return` the answer

The Big Three Recursion Bugs

Bug	Symptom	Fix
Not getting smaller	`RecursionError`	`e - 1`, not `e`
Base case jumped over	`RecursionError`	`<=`, not `==`
Missing `return`	`None` or `TypeError`	`return` the answer

Seen a `RecursionError`? Ask:

does every call get closer to the base case?

Writing Recursive Functions

Factorial

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Factorial

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Factorial

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```
1  def factorial(n):
2      if n <= 1:
3          return 1
4      return n * factorial(n - 1)
5
6  print(factorial(5))
7  print(factorial(0))
```

Factorial

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 \quad = 5 \times 4!$

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```

factorial(5)

Factorial

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5
6  print(factorial(5))
7  print(factorial(0))
```

```
factorial(5)
  = 5 * factorial(4)
```

Factorial

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 \quad = 5 \times 4!$

```
1  def factorial(n):
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5
6  print(factorial(5))
7  print(factorial(0))
```

```
factorial(5)
  = 5 * factorial(4)
    = 5 * 4 * factorial(3)
```

Factorial

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 \quad = 5 \times 4!$

```
1  def factorial(n):
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4      return n * factorial(n - 1)
5
6  print(factorial(5))
7  print(factorial(0))
```

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```

```
factorial(5)
  = 5 * factorial(4)
    = 5 * 4 * factorial(3)
      = 5 * 4 * 3 * factorial(2)
        = 5 * 4 * 3 * 2 * 1
```

= 120

`factorial(0)` is 1.

Length of a String, Without `len`

- The empty string `""` has length **0**. (base case)

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6 print(length("hello"))
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```
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```
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 = 1 + length("ello")
```

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   = 1+1+1 + length("lo")
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   = 1+1+1 + length("lo")
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```

= 5

You Try! *(write the function on paper, 3 min)*

Write `multiply(a, b)` for whole numbers $b \geq 0$. Use **only** +, no $*$.

Hint: $3 \times 4 = 3 + 3 + 3 + 3 = 3 + (3 \times 3)$

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Hint: $3 \times 4 = 3 + 3 + 3 + 3 = 3 + (3 \times 3)$

```
1  def multiply(a, b):
2      if b == 0:
3          return 0
4      return a + multiply(a, b - 1)
5
6  print(multiply(3, 4))      # 12
```

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6  print(multiply(3, 4))      # 12
```

Base case: anything × 0 is 0.

Smaller job: `b - 1`.

The Leap of Faith

To write `factorial(n)`, **trust** that `factorial(n - 1)` already works.

Then ask only: how do I use it to get `factorial(n)`?

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Then ask only: how do I use it to get `factorial(n)`?

Don't trace every call in your head.

Do check the base case, and check that each call is smaller.

Checklist

✓ A **base case**: when to stop.

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- ✓ A **base case**: when to stop.
- ✓ A **recursive case**: call yourself on a smaller job.
- ✓ Every call moves **toward** the base case.
- ✓ Odd inputs (0, negatives, `""`) still stop.
- ✓ Every path ends in a `return`.

You Try! *(write the output, 2 min)*

One `print` **before** the call, one **after**.

(a) What does `echo(2)` print? **(b)** At most, how many `echo` frames?

```
1 def echo(n):
2     if n == 0:
3         print("base")
4         return
5     print("down", n)
6     echo(n - 1)
7     print("up", n)
8
9 echo(2)
```

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9 echo(2)
```

```
down 2
down 1
base
up 1
up 2
```

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down 2
down 1
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```

3 frames

- echo n: 0
- echo n: 1
- echo n: 2
- Global frame

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(b) At most, how many `echo` frames?

```
down 2
down 1
base
up 1
up 2
```

3 frames

- echo n: 0
- echo n: 1
- echo n: 2
- Global frame

Why? The **downs** print going down.

The **ups** print coming back up.

You Try! *(2 min)*

How many times can you halve `n` before you reach 1?

```
1  def halvings(n):
2      if n == 1:
3          return 0
4      return 1 + halvings(n // 2)
```

(a) What does `halvings(8)` return?

(b) What happens with `halvings(0)`?

(c) Fix it. Change **one** line.

You Try! *(2 min)*

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```

```
halvings(8)
= 1 + halvings(4)
= 1 + 1 + halvings(2)
= 1 + 1 + 1 + halvings(1)
= 1 + 1 + 1 + 0 = 3
```

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(b) `0 // 2` is 0. `n` never reaches 1: `RecursionError`.

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```

(b) `0 // 2` is 0. `n` never reaches 1: `RecursionError`.

(c) Line 2: `if n <= 1:`

You Try! *(write the function on paper, 3 min)*

Write `countdown_list(n)`. It **returns** the list `[n, ..., 2, 1]`.

`countdown_list(3)` → `[3, 2, 1]` `countdown_list(0)` → `[]`

Hint: `[3, 2, 1]` is `[3]` + `[2, 1]`

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```
1 def countdown_list(n):
2     if n == 0:
3         return []
4     return [n] + countdown_list(n - 1)
5
6 print(countdown_list(3))
```

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2     if n == 0:
3         return []
4     return [n] + countdown_list(n - 1)
5
6 print(countdown_list(3))
```

Base case: 0 gives the empty list `[]`.

Smaller job: `n - 1`.

Summary

Summary

- **Recursion:** a function calls itself on a smaller job.
- **Two parts:** a base case and a recursive case.
- **Call stack:** every call gets a frame. The newest is on top.
- **No base case:** about 1000 frames, then `RecursionError`.
- **Big Three bugs:** not smaller, base case jumped over, missing `return`.

Big Idea

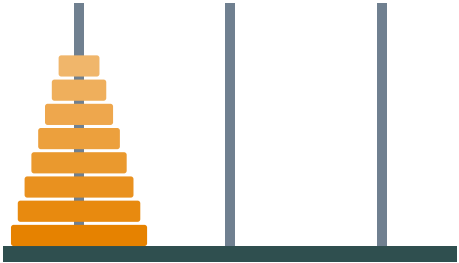
A function may call itself...

if each call is **smaller**
and a **base case** stops it.

Without a base case, the frames pile up until they **overflow the stack**.

Next Time: Thinking Recursively

- A temple. Three needles. **64** golden disks.



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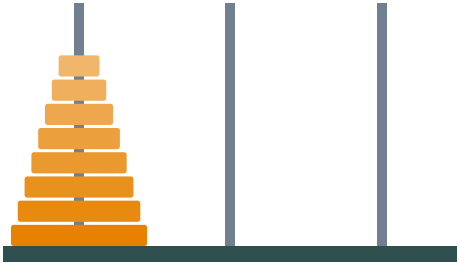

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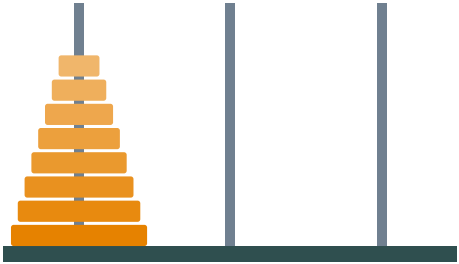

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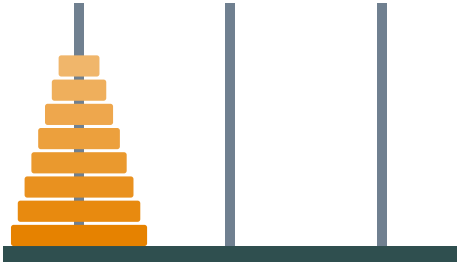

Should we worry? Recursion will tell us.

Questions?

Exit Ticket

Today's question *(2 minutes)*

(a) What does `f(3)` return?

```
def f(n):
    if n == 0:
        return 0
    return n * 2 + f(n - 1)
```

(b) Why is the website called **Stack Overflow**? One line.

COMP 102 — Exit Ticket #21

Name: _____

Roll #: _____ Date: _____

Answer:

Hand it to me on your way out.
No name, no attendance.